

Generic wormhole throats

Matt Visser[†] and David Hochberg[‡]

[†]Physics Department, Washington University
Saint Louis, Missouri 63130-4899, USA

[‡]Laboratorio de Astrofísica Espacial y Física Fundamental
Apartado 50727, 28080 Madrid, Spain

15 October 1997

[†] e-mail: visser@kiwi.wustl.edu

[‡] e-mail: hochberg@laeff.esa.es

gr-qc/9710001

To appear in the proceedings of the Haifa Workshop:
The Internal Structure of Black Holes
and Spacetime Singularities.

Abstract

Wormholes and black holes have, apart from a few historical oddities, traditionally been treated as quite separate objects with relatively little overlap. The possibility of a connection arises in that wormholes, if they exist, might have profound influence on black holes, their event horizons, and their internal structure. For instance: (1) small wormhole-induced perturbations in the geometry can lead to massive non-perturbative shifts in the event horizon, (2) Planck-scale wormholes near any spacelike singularity might let information travel in effectively spacelike directions, (3) vacuum polarization effects near any singularity might conceivably lead to a “punch through” into another asymptotically flat region—effectively transforming a black hole into a wormhole. After discussing these connections between black hole physics and wormhole physics we embark on an overview of what can generally be said about traversable wormholes and their throats. We discuss the violations of the energy conditions that typically occur at and near the throat of any traversable wormhole and emphasize the generic nature of this result. We discuss the original Morris–Thorne wormhole and its generalization to a spherically symmetric

time-dependent wormhole. We discuss spherically symmetric Brans–Dicke wormholes as examples of how to hide the energy condition violations in an inappropriate choice of definitions. We also discuss the relationship of these results to the topological censorship theorem. Finally we turn to a rather general class of wormholes that permit explicit analysis: generic static traversable wormholes (without any symmetry). We show that topology is too limited a tool to accurately characterize a generic traversable wormhole—in general one needs geometric information to detect the presence of a wormhole, or more precisely to locate the wormhole throat. For an arbitrary static spacetime we shall define the wormhole throat in terms of a 2-dimensional constant-time hypersurface of minimal area. (Zero trace for the extrinsic curvature plus a “flare-out” condition.) This enables us to severely constrain the geometry of spacetime at the wormhole throat and to derive generalized theorems regarding violations of the energy conditions—theorems that do not involve geodesic averaging but nevertheless apply to situations much more general than the spherically symmetric Morris–Thorne traversable wormhole. [For example: the null energy condition (NEC), when suitably weighted and integrated over the wormhole throat, must be violated.]

1 Introduction

Traversable wormholes [1, 2, 3] have traditionally been viewed as quite distinct from black holes, with essentially zero overlap in techniques and topics. (Historical oddities that might at first seem to be counter-examples to the above are the Schwarzschild wormhole, which is not traversable, and the Einstein–Rosen bridge, which is simply a bad choice of coordinates on Schwarzschild spacetime [3, pages 45–51].) Since this workshop is primarily directed toward the study of black holes we shall start by indicating some aspects of commonality and inter-linkage between these objects.

First: We point out that traversable wormholes, if they exist, can have violent non-perturbative influence on black hole event horizons. This is fundamentally due to the fact that the event horizon is defined in a global manner, so that small changes in the geometry, because they have all the time in the universe to propagate to future null infinity, can have large effects on the event horizon.

Second: We point out that the region near any curvature singularity is expected to be subject to large fluctuations in the metric. If Wheeler’s spacetime foam picture is to be believed, the region near a curvature singularity should be infested with wormholes. If it’s a spacelike singularity (meaning, it’s spacelike before you take the wormholes into account) then the wormholes can connect regions that would otherwise be outside each other’s lightcones, and so lead to information transport in an effectively spacelike direction. You still have to worry about how to transfer the information across the event horizon, but this

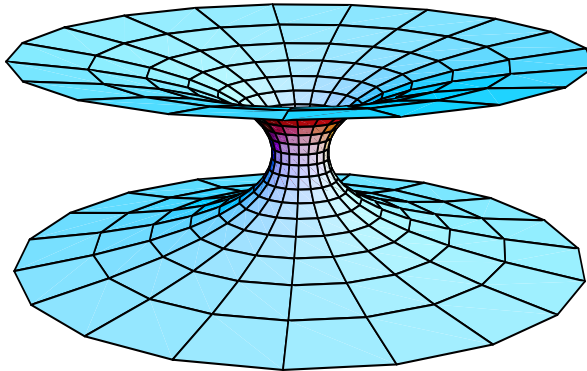


Figure 1: Schematic representation of an inter-universe wormhole connecting two asymptotically flat regions.

is a start towards a lazy way out of the black hole information paradox. (The information paradox is only a paradox if you take the Schwarzschild singularity too seriously, in particular if you take the spacelike nature of the singularity too seriously.)

Third: In addition to the fluctuations that take place near a curvature singularity, we would also expect large effects on the expectation value of the metric due to gravitational vacuum polarization. Since gravitational vacuum polarization typically violates the energy conditions, this could lead us to expect a “punch through” to another asymptotically flat region. The possibility that singularities might heal themselves by automatic conversion into wormhole throats, though maybe unlikely, should at least be kept in mind.

After developing these linkages between black hole physics and wormhole physics, we embark on an overview of traversable wormholes, concentrating on the region near the throat of the wormhole. One of the most important features that characterizes traversable wormholes is the violation of the energy conditions, in particular the null energy condition, *at or near the throat*. The energy condition violations were first discovered in the static spherically symmetric Morris–Thorne wormholes, but the result is generic (modulo certain technical assumptions) as borne out by the topological censorship theorem, and also by the generic analysis of static wormholes developed later in this survey.

We shall discuss the case of time-dependent spherically symmetric wormholes, wherein the energy violation conditions can be isolated at particular re-

gions in time (in the same way that static thin-shell wormholes permit one to isolate the energy condition violations at particular regions of space.) We show that these results are compatible with the topological censorship theorem. Perhaps surprisingly, we show that cosmological inflation is useless in terms of generating even temporary suspension of the energy condition violations.

We further discuss the case of spherically symmetric Brans–Dicke wormholes as an example of what happens in non–Einstein theories of gravity: In this case it is possible to mistakenly conclude that the energy conditions are not violated, but this would merely be a consequence of an inappropriate choice of definitions. For instance: If one works in the Einstein frame, defines the existence of a wormhole in terms of the Einstein metric, and calculates using the total stress energy tensor, then the null energy condition must be violated at or near the throat. If one artificially divides the total stress-energy into (Brans–Dicke stress-energy) plus (ordinary stress-energy), then since the Brans–Dicke stress energy (calculated in the Einstein frame) sometimes violates the energy conditions (if the Brans–Dicke parameter ω is less than $-3/2$), the “ordinary” part stress-energy can sometimes satisfy the null energy condition.

Similarly, suppose one works in the Jordan frame, and defines the existence of a wormhole using the Jordan metric. If one calculates the total stress energy tensor then it is still true that the null energy condition must be violated at or near the throat. If one now artificially divides the total stress-energy into (Brans–Dicke stress-energy) plus (ordinary stress-energy), then for suitable choices of the Brans–Dicke parameter ω ($\omega < -2$) one can hide all the energy condition violations in the Brans–Dicke field and permit the ordinary stress-energy to satisfy the energy conditions.

To further confuse the issue, one could define the existence (or nonexistence) of a wormhole using one frame, and then calculate the Einstein tensor and total stress-energy in the other frame. This is a dangerous and misleading procedure: We shall show that the definition of the existence and location of a wormhole throat is not frame independent (because it is not conformally invariant). Jumping from one frame to the other in the middle of the calculation can easily lead to meaningless results. It is critical to realise that the energy condition violations must still be there and in fact are still there: they have merely been hidden by sleight of hand. Qualitatively, these comments also apply to other non–Einstein theories of gravity such as the Einstein–Cartan theory, Dilaton gravity, Lovelock gravity, Gauss–Bonnet gravity, etc...

Finally, we wrap up by presenting a general analysis for static traversable wormholes that completely avoids all symmetry requirements and even avoids the need to assume the existence of asymptotically flat regions. This exercise is particularly useful in that it places the notion of wormhole in a much more general setting. Indeed, wormholes are often viewed as intrinsically topological objects, occurring in multiply connected spacetimes. The Morris–Thorne class of inter-universe traversable wormholes is even more restricted, requiring both exact spherical symmetry and the existence of two asymptotically flat regions in

the spacetime. To deal with intra-universe traversable wormholes, the Morris–Thorne analysis must be subjected to an approximation procedure wherein the two ends of the wormhole are distorted and forced to reside in the same asymptotically flat region. The existence of one or more asymptotically flat regions is an essential ingredient of the Morris–Thorne approach [1].

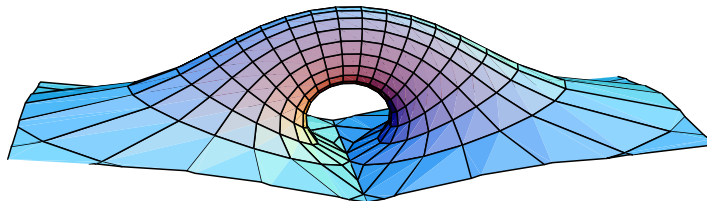


Figure 2: Schematic representation of an intra-universe wormhole formed by deforming an inter-universe wormhole and forcing the two asymptotically flat regions to merge.

However, there are many other classes of geometries that one might still quite reasonably want to classify as wormholes, that either have trivial topology [3], or do not possess any asymptotically flat region [4], or exhibit both these phenomena.

A simple example of a wormhole lacking an asymptotically flat region is two closed Friedman–Robertson–Walker spacetimes connected by a narrow neck (see figure 3), you might want to call this a “dumbbell wormhole”. A simple example of a wormhole with trivial topology is a single closed Friedman–Robertson–Walker spacetime connected by a narrow neck to ordinary Minkowski space (see figure 4). A general taxonomy of wormhole exemplars may be found in [3, pages 89–93], and discussions of wormholes with trivial topology may also be found in [3, pages 53–74].

To set up the analysis for a generic static throat, we first have to define exactly what we mean by a wormhole—we find that there is a nice *geometrical* (not topological) characterization of the existence of, and location of, a wormhole “throat”. This characterization is developed in terms of a hypersurface of minimal area, subject to a “flare-out” condition that generalizes that of the Morris–Thorne analysis.

With this definition in place, we can develop a number of theorems about the existence of “exotic matter” at the wormhole throat. These theorems generalize the original Morris–Thorne result by showing that the null energy condition (NEC) is generically violated at some points on or near the two-dimensional surface comprising the wormhole throat. These results should be viewed as complementary to the topological censorship theorem [5]. The topological cen-

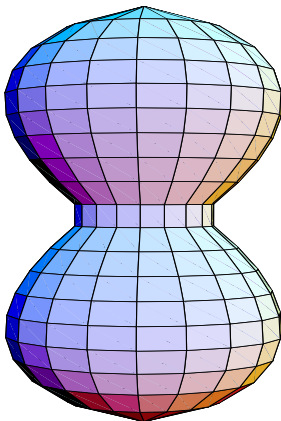


Figure 3: A “dumbbell wormhole”: Formed (for example) by two closed Friedman–Robertson–Walker spacetimes connected by a narrow neck.

sorship theorem tells us that in a spacetime containing a traversable wormhole the averaged null energy condition must be violated along at least some (not all) null geodesics, but the theorem provides very limited information on where these violations occur. The analysis of this paper shows that some of these violations of the energy conditions are concentrated in the expected place: on or near the throat of the wormhole. The present analysis, because it is purely local, also does not need the many technical assumptions about asymptotic flatness, future and past null infinities, and global hyperbolicity that are needed as ingredients for the topological censorship theorem [5].

The key simplifying assumption in the present analysis is that of taking a static wormhole. While we believe that a generalization to dynamic wormholes is possible, the situation becomes technically much more complex and one is rapidly lost in an impenetrable thicket of definitional subtleties and formalism. (A suggestion, due to Page, whereby a wormhole throat is viewed as an *anti-trapped surface* in spacetime holds promise for suitable generalization to the fully dynamic case [6].)

In summary, the violations of the energy conditions at wormhole throats are unavoidable. Many of the attempts made at building a wormhole without violating the energy conditions do so only by hiding the energy condition violations in some subsidiary field, or by hiding the violations at late or early times.

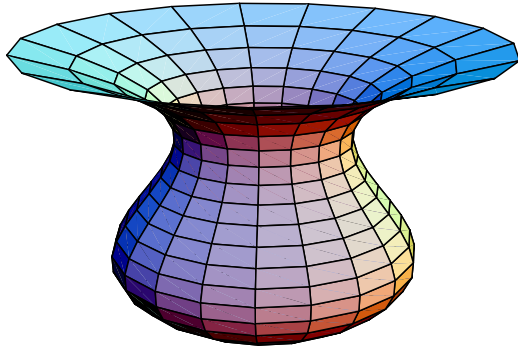


Figure 4: A wormhole with trivial topology: Formed (for example) by connecting a single closed Friedman–Robertson–Walker spacetime to Minkowski space by a narrow neck.

2 Connections

We start by describing a few scenarios whereby wormholes might prove of interest to black hole physics. Don't take any of these scenarios too seriously: they are presented more as suggestions for things to consider than as definite proposals for serious models.

2.1 Non-perturbative changes in the event horizon

Traversable wormholes, if they exist, can lead to massive nonperturbative changes in the event horizon of a black hole at the cost of relatively minor perturbations in the geometry. (This should not be too surprising: it is a general feature of Einstein gravity that small perturbations in the geometry can lead to massive perturbations in the event horizon.) One of the simplest examples of this effect is to take a wormhole (with two mouths), and a black hole, and then throw one wormhole mouth down into the black hole while keeping the second wormhole mouth outside [7]. Everything in the past lightcone of the mouth that falls into the wormhole, including the segment that is behind where the event horizon used to be before the wormhole mouth fell in, can now influence future null infinity by taking a shortcut through the wormhole.

This can best be seen visually; we idealize the wormhole to have a mouth extremely small radius, and model the wormhole by a pair of timelike lines in spacetime that are mathematically identified. The black hole will be idealized in the usual way with a Penrose diagram. The geometry of the spacetime is affected by the wormhole only in the immediate vicinity of the wormhole mouths, but the global properties (such as the event horizon) suffer drastic non-perturbative changes.

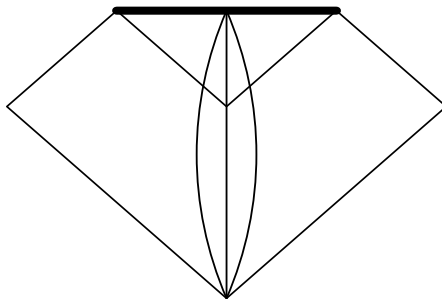


Figure 5: Penrose diagram for an astrophysical black hole formed by (say) stellar collapse.

With a swarm of traversable wormholes one can chip away at the event horizon to the extent that it “almost” disappears. The event horizon is progressively eaten away as more wormhole mouths fall into the region behind the apparent horizon.

2.2 Effectively spacelike information transfer

Near any spacetime curvature singularity, in the region where curvatures are large, the spacetime foam picture developed by Wheeler [8, 9] strongly suggests that Lorentzian wormholes will be rapidly popping in and out of existence. (We ignore for the sake of present discussion potential difficulties associated with topology change in Lorentzian manifolds: certainly something peculiar has to happen in the region where the wormhole is created or destroyed. See [3, pages 61–73].)

We can think of at least two plausible models for Lorentzian wormhole creation. (We again take the approximation that the wormholes have infinitely small mouth radius, and so can be modelled by infinitely thin spacetime world-lines.) In model (1) the two mouths are created (pulled out of the Planck slop?) at different places, and after creation one simply has two timelike world-lines that are to be identified. In model (2) the two mouths are created at the same spacetime point and the two mouths then subsequently move off along distinct

timelike world-lines (which are again identified to produce the wormhole structure).

In model (1) the spacetime distribution of the creation points for the two mouths is something we have no idea how to calculate. We should hope that the two creation points are spacelike separated since otherwise the Chronology Protection Conjecture [10] has failed at the outset. Beyond that, very little can be said. In model (2) the Chronology Protection Conjecture is respected at the initial creation point, but one still has to worry about the subsequent evolution of the wormhole mouths. Since all of this is presumably taking place deep inside the Planck slop, issues of reliability of the entire semiclassical approximation also deserve attention [11].

Whichever of these models (1) or (2) you pick [and both models are perfectly compatible with both the typical noises generated in this field, and our current ignorance of quantum gravity] the wormhole mechanism takes some points that would be spacelike separated if the wormholes were not present and makes them timelike separated. This is quite sufficient to allow “effectively spacelike” information transfer in a thin region near the curvature singularity.

To actually get information out of the black hole, you will either have to live with an extension of model (1) wherein the second wormhole mouth pops into existence just outside the event horizon, or rely on an external wormhole pair one of whose mouths is permitted to fall into the horizon.

2.3 “Punch-through” near the singularity

Near a spacelike curvature singularity, the quantum vacuum expectation value of the stress-energy tensor is likely to be heading off to infinity due to gravitational vacuum polarization effects. But gravitational vacuum polarization quite typically leads to violations of the energy conditions [12, 13, 14, 15, 16]. And violations of the energy conditions are a generic feature of wormholes.

This suggests (hints) that curvature singularities might heal themselves by “punching through” to another asymptotically flat region in a manner similar to a wormhole. (See for example the minisuperspace model discussed in [3, pages 347–359] and [17, 18, 19].) For a concrete suggestion along these lines consider the metric

$$ds^2 = -\left(1 - \frac{2GM}{\sqrt{\ell^2 + r^2}}\right) dt^2 + \left(1 - \frac{2GM}{\sqrt{\ell^2 + r^2}}\right)^{-1} dr^2 + (\ell^2 + r^2) \{d\theta^2 + \sin^2 \theta d\phi^2\}. \quad (1)$$

This is almost the Schwarzschild geometry, apart from the parameter ℓ . The radial variable r can now be extended all the way from $+\infty$ to $-\infty$, and there is a symmetry under interchange $r \rightarrow -r$. For $\ell \ll 2GM$ (but $\ell \neq 0$) and r not too close to zero (which is where the spacelike singularity is for $\ell = 0$), this is

indistinguishable from the Schwarzschild solution. There is a thin region near $r = 0$ where the stress-energy is appreciably different from zero. In fact the Einstein tensor is

$$G_{\hat{t}\hat{t}} = -\frac{\ell^2 \left(1 - \frac{4GM}{\sqrt{\ell^2 + r^2}}\right)}{(\ell^2 + r^2)^2}, \quad (2)$$

$$G_{\hat{\theta}\hat{\theta}} = +\frac{\ell^2 \left(1 - \frac{GM}{\sqrt{\ell^2 + r^2}}\right)}{(\ell^2 + r^2)^2}, \quad (3)$$

$$G_{\hat{r}\hat{r}} = -\frac{\ell^2}{(\ell^2 + r^2)^2}, \quad (4)$$

Note that this is not a traversable wormhole in the usual sense, since the region near the “throat” ($r = 0$) it is the radial direction that is timelike. Thus travel is definitely one-way, and if anything one should call this a “timehole”. The hope of course is that some sort of geometry similar to the above might emerge as a self-consistent solution to the semiclassical field equations. A time-like version of the Hochberg–Popov–Sushkov analysis [4] is what we have in mind. Note in particular that at $r = 0$ we have

$$G_{\hat{t}\hat{t}} + G_{\hat{r}\hat{r}} = +\frac{(4GM - 2\ell)}{\ell^3}. \quad (5)$$

Thus the null energy condition is *not* violated at $r = 0$, indicating that one-way “timeholes” of this type are qualitatively different from the two-way traversable wormholes considered in the rest of this survey. (There are some similarities here with the Aichelburg–Schein wormholes, which are not true traversable wormholes in that they are only one-way traversable [20]. In contrast the Aichelburg–Israel–Schein wormholes are true traversable wormholes which violate the energy conditions in the usual manner [21].)

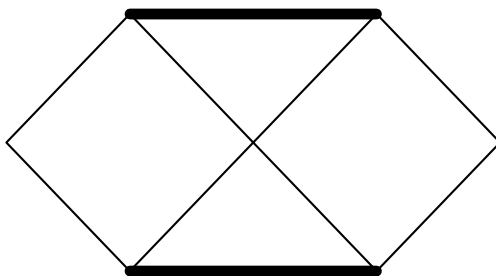


Figure 6: Penrose diagram for the maximally extended Schwarzschild geometry.

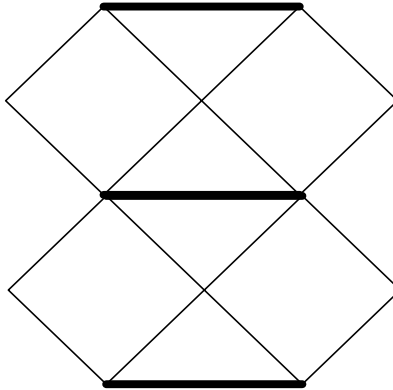


Figure 7: Penrose diagram for a “timehole”. Gravitational vacuum polarization near where the spacelike singularity would have been could plausibly lead to “punch through” to another asymptotically flat region.

2.4 “Punch-through” near the horizon

A bolder proposal is that “punch through” might occur not near where the central singularity would have been, but instead that “punch through” might occur out where one would naively expect the event horizon to be. For instance, if one insists on using the Boulware vacuum, and insists on a spacetime containing an event horizon, then the renormalized stress-energy will diverge at the horizon. Therefore, if one insists on having a self-consistent solution to the semiclassical field equations in the Boulware vacuum, then no event horizon can survive. Given that test-field calculations on Schwarzschild spacetime show infinite gravitational vacuum polarization at the would-be event horizon, and that this gravitational vacuum polarization violates the null energy condition, one might suspect that the self-consistent solution contains a traversable wormhole.

Bolstering this suspicion is the fact that the Hochberg–Popov–Sushkov analysis [4] has numerically integrated the fourth-order differential equations derived from semiclassical gravity, and found wormhole throats in the Boulware vacuum. (There are a tangle of technical issues here to do with the size of the wormhole throat and whether or not it is possible to extend these numerical solutions out into truly asymptotically flat regions. We refer the interested reader to the literature.)

A simple and explicit toy model geometry along these lines is to start with the Schwarzschild solution written in isotropic coordinates

$$ds^2 = - \left(\frac{1 - GM/2r}{1 + GM/2r} \right)^2 dt^2$$

$$+(1 + GM/2r)^4 \{dr^2 + r^2[d\theta^2 + \sin^2 d\phi^2]\}. \quad (6)$$

The radial coordinate r runs from $r = 0$ to $r = +\infty$. The horizon is at $r = R \equiv GM/2$, and there is symmetry under radial inversion $r \rightarrow R^2/r$. Now perturb this geometry by distorting the g_{tt} component of the metric:

$$ds^2 = - \left\{ \frac{(R^2 - r^2)^2 + \ell^2 r^2}{(R + r)^4 + \ell^2 r^2} \right\} dt^2 + (1 + R/r)^4 \{dr^2 + r^2[d\theta^2 + \sin^2 d\phi^2]\}. \quad (7)$$

The radial coordinate r still runs from $r = 0$ to $r = +\infty$ and there is still a symmetry under radial inversion $r \rightarrow R^2/r$. The particular form of the distortion given above has been chosen so that the asymptotic form of the metric at spatial infinity agrees with the Schwarzschild metric up to order $O[r^{-3}]$. But $r = R \equiv GM/2$ is no longer an event horizon, it is now the location of a wormhole throat connecting two asymptotically flat regions (one at $r = +\infty$ and the other at $r = 0$). The throat is at finite redshift $1 + z = \sqrt{1 + (16R^2/\ell^2)}$. For $\ell \ll R$ the region outside the would-be event horizon is arbitrarily close to Schwarzschild geometry. (This is an example of a “proximal Schwarzschild” wormhole as discussed in [3, pages 147–149].) The stress-energy tensor is a bit messy, but falls off as order $O[r^{-5}]$ for large r . Thus there is a thin layer of energy condition violating matter near the throat of the wormhole, but by the time you are a little distance away from the throat you cannot tell the difference between this geometry and the usual Schwarzschild solution.

2.5 Summary

The key point to be extracted from the above discussion is that small perturbations on the geometry of spacetime can make drastic perturbations to the notion of a black hole. Small perturbations of the geometry can eat away at the event horizon like cancer, can drastically affect the properties of spacelike singularities (spacelike information transfer, time holes), and in the wrong hands can even make event horizons vanish (proximal Schwarzschild wormholes). All of the specific examples discussed above use wormholes, or things that are almost wormholes, so we feel it is a good idea for black hole physicists to have some basic understanding of the wormhole system and its limitations.

3 Energy conditions: An overview

The Morris–Thorne analysis [1] revitalized interest in Lorentzian traversable wormholes. Morris and Thorne were able to show that traversable wormholes were compatible with our current understanding of general relativity and semi-classical quantum gravity — but that there was a definite price to be paid — one had to admit violations of the null energy condition.

More precisely, what Morris and Thorne showed was equivalent to the statement that for spherically symmetric traversable wormholes there must be an open region surrounding the throat over which the null energy condition is violated [1, 3]. The striking nature of this result has led numerous authors to try to find ways of evading or minimizing the energy condition violations, and on the other hand has led to a number of general theorems guaranteeing the existence of these violations.

For instance, static but not spherically symmetric thin-shell wormholes and their variants allow you to move the energy condition violations around in space, so that there are some routes through the wormhole that do not encounter energy condition violations. (This is most easily seen using “cut and paste” wormholes constructed using the thin shell formalism [22]. See also [3, pages 153–194].)

Time dependent but spherically symmetric wormholes allow you to move the energy condition violations around in time [23, 24, 25, 26, 27]. (Unfortunately radial null geodesics through the wormhole will still encounter energy condition violations, subject to suitable technical qualifications.)

The Friedman–Schleich–Witt topological censorship theorem [5], under suitable technical conditions, guarantees that spacetimes containing traversable wormholes must contain some null geodesics that violate the averaged null energy condition (but gives little information on where these null geodesics must be).

Attempts at eliminating the energy condition violations completely typically focus on alternative gravity theories (Brans–Dicke gravity, Dilaton gravity, gravity with torsion). We argue that such attempts are at best sleight of hand—it is sometimes possible to hide the energy condition violations in the Brans–Dicke field, or the dilaton field, or the torsion, but the energy condition violations are still always there.

3.1 Morris–Thorne wormhole

Take any spherically symmetric static spacetime geometry and use the radial proper distance as the radial coordinate. The metric can then without loss of generality be written as

$$ds^2 = -e^{2\phi(l)} dt^2 + dl^2 + r^2(l) [d\theta^2 + \sin^2 \theta d\varphi^2]. \quad (8)$$

If we wish this geometry to represent a Morris–Thorne wormhole then we must impose conditions both on the throat and on asymptotic infinity [1, 3].

- Conditions at the throat:
 - The absence of event horizons implies that $\phi(l)$ is everywhere finite.
 - The radius of the wormhole throat is defined by

$$r_0 = \min\{r(l)\}. \quad (9)$$

- For simplicity one may assume that there is only one such minimum and that it is an isolated minimum. Generalizing this point is straightforward.
- Without loss of generality, we can take this throat to occur at $l = 0$.
- The metric components should be at least twice differentiable as functions of l .
- Conditions at asymptotic infinity:
 - The coordinate l covers the entire range $(-\infty, +\infty)$.
 - There are two asymptotically flat regions, at $l \approx \pm\infty$.
 - In order for the spatial geometry to tend to an appropriate asymptotically flat limit we impose

$$\lim_{l \rightarrow \pm\infty} \{r(l)/|l|\} = 1. \quad (10)$$

- In order for the spacetime geometry to tend to an appropriate asymptotically flat limit, we impose finite limits

$$\lim_{l \rightarrow \pm\infty} \phi(l) = \phi_{\pm}. \quad (11)$$

- These are merely the minimal requirements to obtain a wormhole that is “traversable in principle”. For realistic models, “traversable in practice”, one should address additional engineering issues such as tidal effects [3, pages 137–152].
- We shall argue, later in this survey, that the conditions imposed at asymptotic infinity can be relaxed (and for certain questions, asymptotic infinity can be ignored completely), and that it is the throat of the wormhole that is often more of direct interest.

It is an easy exercise to show that the Einstein tensor is [3, pages 149-150]

$$G_{\hat{t}\hat{t}} = -\frac{2r''}{r} + \frac{1 - (r')^2}{r^2}, \quad (12)$$

$$G_{\hat{r}\hat{r}} = \frac{2\phi' r'}{r} - \frac{1 - (r')^2}{r^2}, \quad (13)$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\varphi}\hat{\varphi}} = \phi'' + (\phi')^2 + \frac{\phi' r' + r''}{r}. \quad (14)$$

Thus in particular

$$G_{\hat{t}\hat{t}} + G_{\hat{r}\hat{r}} = -\frac{2r''}{r} + \frac{2\phi' r'}{r}. \quad (15)$$

But by definition $r' = 0$ at the throat. We also know that $r'' \geq 0$ at the throat. The possibility that $r'' = 0$ at the throat forces us to invoke some technical complications. By definition the throat is a local minimum of $r(l)$. Thus there must be an open region $l \in (0, l_*^+)$ such that $r''(l) > 0$, and on the other side of the throat an open region $l \in (-l_*^-, 0)$ such that $r''(l) > 0$. This is the Morris–Thorne “flare-out” condition. So for the Einstein tensor we have

$$\exists l_*^-, l_*^+ > 0 : \quad \forall l \in (-l_*^-, 0) \cup (0, l_*^+), \quad G_{\hat{t}\hat{t}} + G_{\hat{r}\hat{r}} < 0. \quad (16)$$

This constraint on the components of the Einstein tensor follows directly from the definition of a traversable wormhole and the definition of the Einstein tensor—it makes no reference to the dynamics of general relativity and automatically holds *by definition* regardless of whether one is dealing with Einstein gravity, Brans–Dicke gravity, or any other exotic form of gravity. As long as you have a spacetime metric that contains a wormhole, and calculate the Einstein tensor using that *same* spacetime metric, the above inequality holds by definition.

We can always define the total stress energy by enforcing the Einstein equations

$$G^{\mu\nu} = 8\pi G T_{total}^{\mu\nu}. \quad (17)$$

In terms of the total stress energy

$$\exists l_*^-, l_*^+ > 0 : \quad \forall l \in (-l_*^-, 0) \cup (0, l_*^+), \quad T_{\hat{t}\hat{t}}^{total} + T_{\hat{r}\hat{r}}^{total} < 0. \quad (18)$$

Thus the null energy condition for the total stress energy must be violated on some open region surrounding the throat. The only requirements for this result are essentially matters of definition: use the same metric to define the wormhole and to calculate the Einstein tensor, and use that same Einstein tensor to identify the total stress-energy.

(This also shows where the maneuvering room is in exotic theories of gravity. If you have multiple metrics, multiple Einstein tensors, and multiple definitions of stress-energy then it becomes easy to hide the violations of the energy conditions in inappropriate definitions.)

3.2 Spherically symmetric time-dependent wormholes

After the initial treatment by Morris and Thorne it was quickly realized that by eschewing spherical symmetry it is quite possible to minimize the violations of the null energy condition [22]. In particular it is quite possible to move the regions subject to energy condition violations around in space so as to hide them in the woodwork and permit at least some travelers through the wormhole completely avoid any personal contact with exotic matter.

Somewhat later, it was realized that *time dependence* lets one move the energy condition violating regions around in time [25, 26], and so leads to a temporary suspension of the need for energy condition violations. (For related analyses see also [23, 24, 27].) The most direct presentation of the key results can best be exhibited by taking a spacetime metric that is conformally related to a zero-tidal force wormhole by a simple time-dependent but space-independent conformal factor. Thus

$$ds^2 = \Omega(t)^2 \left\{ -dt^2 + \frac{dr^2}{1-b(r)/r} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\}. \quad (19)$$

(This form of the metric has the advantage that null geodesics remain null geodesics as Ω is altered.) It is an easy exercise to see that

$$G_{\hat{t}\hat{t}} = +\Omega^{-2} \left(\frac{b'(r)}{r^2} + \frac{\dot{\Omega}^2}{\Omega^2} \right), \quad (20)$$

$$G_{\hat{\theta}\hat{\theta}} = +\Omega^{-2} \left(\frac{b(r)}{2r^3} - \frac{b'(r)}{2r^3} + \frac{\dot{\Omega}^2}{\Omega^2} - 2\frac{\ddot{\Omega}}{\Omega} \right), \quad (21)$$

$$G_{\hat{r}\hat{r}} = -\Omega^{-2} \left(\frac{b(r)}{r^3} + 2\frac{\ddot{\Omega}}{\Omega} - \frac{\dot{\Omega}^2}{\Omega^2} \right). \quad (22)$$

In particular, looking along the radial null direction

$$G_{\hat{t}\hat{t}} + G_{\hat{r}\hat{r}} = \Omega^{-2} \left(-\frac{b(r)}{r^3} + \frac{b'(r)}{r^3} 2 - 2\frac{\ddot{\Omega}}{\Omega} + 4\frac{\dot{\Omega}^2}{\Omega^2} \right) \quad (23)$$

We can rewrite this in terms of the Hubble parameter H and deceleration parameter q , by first relating conformal time t to comoving time T via $\Omega dt = dT$. We then deduce

$$H = \Omega^{-1} \frac{d\Omega}{dT} = \Omega^{-2} \dot{\Omega}, \quad (24)$$

and

$$q = -\Omega \frac{d^2\Omega}{dT^2} \left(\frac{d\Omega}{dT} \right)^{-2} = 1 - \frac{\Omega \ddot{\Omega}}{\dot{\Omega}^2}. \quad (25)$$

Then

$$G_{\hat{t}\hat{t}} + G_{\hat{r}\hat{r}} = \Omega^{-2} \left(-\frac{b(r)}{r^3} + \frac{b'(r)}{r^3} \right) + 2H^2(1+q). \quad (26)$$

So if the universe expands quickly enough we can (temporarily) suspend the violations of the null energy condition. Note that the first term above is just

the static $\Omega = 1$ result rescaled to the expanded universe. At the throat of the wormhole this is of order $-\xi/a^2$ where a is the physical size of the wormhole mouth and ξ is a dimensionless number typically of order unity unless some fine tuning is envisaged. This implies that the expansion of the universe can overcome the violations of the energy condition only if

$$a > \sqrt{\frac{\xi}{1+q}} \frac{1}{H}. \quad (27)$$

This requires a wormhole throat with radius of order the Hubble distance or larger (modulo possible fine tuning). This indicates that suspension of the energy condition violations may be of some interest during the very early universe when the Hubble parameter is large, but that in the present epoch it would be quite impractical to rely on the expansion of the universe to avoid the need for energy condition violations.

It is perhaps surprising to realise that the inflationary epoch, though it can be invoked to make wormholes larger by inflating them out of the Planck slop (Wheeler's spacetime foam) up to more manageable sizes [23, 24], cannot be relied upon to aid in the suspension of energy condition violations. This arises because $q = -1$ during the inflationary epoch so that for the metric considered above

$$G_{\hat{t}\hat{t}} + G_{\hat{r}\hat{r}} = \Omega^{-2} \left(-\frac{b(r)}{r^3} + \frac{b'(r)}{r^3} \right). \quad (28)$$

This should be compared to equation (3.8) of [23], which was obtained directly in terms of a comoving time formalism assuming inflationary expansion from the outset. In the formalism developed above, this can be checked by noting that during the inflationary epoch

$$\Omega = \exp(HT) = \frac{1}{1 - Ht}, \quad (29)$$

where we have normalized $t = 0$ at $T = 0$. It is then easy to explicitly check that $q = -1$.

If we want to compare this result with that of the topological censorship theorem [5] it is important to realise that the topological censorship theorem applies to the current situation only if we switch off the expansion of the universe at sufficiently early and late times. This is because the topological censorship theorem uses asymptotic flatness, in both space and time, as an essential ingredient in setting up both the statement of the theorem and the proof. (Scri^- , past null infinity, simply makes no sense in a big bang spacetime.)

So, provided we switch off the expansion of the universe at sufficiently early and late times, we deduce first that null energy condition violations reappear at sufficiently early and late times, and more stringently, that radial null geodesics must violate the ANEC.

In brief, the suspension of the energy condition violations afforded by time dependence are either transitory or intimately linked to the existence of a big bang singularity, and in either case are most likely limited to small scale microscopic wormholes in the pre-inflationary epoch.

3.3 Brans–Dicke wormholes

Brans–Dicke wormholes [28, 29, 30] are particular examples of the effect that choosing a non-Einstein model for gravity has on the behaviour of traversable wormholes. The Brans–Dicke theory of gravity is perhaps the least violent alteration to Einstein gravity that can be contemplated. In the Brans–Dicke theory the gravitational field is composed of two components: a spacetime metric plus a dynamical scalar field. (A word of caution: some papers dealing with Brans–Dicke wormholes are actually in Euclidean signature [31]. Euclidean wormholes are qualitatively different from Lorentzian wormholes and will not be discussed in this survey.)

3.3.1 The Jordan Frame

In the so-called Jordan frame the Action is [36, page 1070]

$$S = \int \sqrt{-g} \left\{ \phi R - \omega \frac{(\nabla \phi)^2}{\phi} + 16\pi G \mathcal{L}_{matter} \right\}. \quad (30)$$

The equations of motion are

$$G_{\alpha\beta} = \frac{8\pi G}{\phi} T_{\alpha\beta}^{matter} + \frac{\omega}{\phi^2} \left\{ \phi_\alpha \phi_\beta - \frac{1}{2} g_{\alpha\beta} (\nabla \phi)^2 \right\} + \frac{1}{\phi} \{ \phi_{;\alpha\beta} - g_{\alpha\beta} \nabla^2 \phi \}, \quad (31)$$

and

$$\nabla^2 \phi = \frac{8\pi G}{3 + 2\omega} T. \quad (32)$$

If we wish the metric g to describe a spherically symmetric static wormhole, then the Morris–Thorne analysis implies that the total stress energy defined by $T_{\alpha\beta}^{total} = G_{\alpha\beta}/(8\pi G)$ must violate the null energy condition. Whether or not the “matter” part of the total stress-energy violates the null energy condition depends on how the Brans–Dicke scalar field behaves at and near the throat.

If we assume that the wormholes we are looking for have $\phi \neq 0$ and $\phi \neq \infty$ at the throat, then for any radial null vector k^α we have

$$G_{\alpha\beta} k^\alpha k^\beta = \frac{8\pi G}{\phi} T_{\alpha\beta}^{matter} k^\alpha k^\beta + \omega \frac{(k^\alpha \phi_\alpha)^2}{\phi^2} + \frac{\phi_{;\alpha\beta} k^\alpha k^\beta}{\phi} \leq 0. \quad (33)$$

This implies that the only way in which the “matter” part of the stress-energy can avoid violating the null energy condition is if either $\omega < 0$ and $\nabla\phi \neq 0$ at the throat, or if ϕ is convex at the throat. We now verify these general conclusions by looking at some specific exact solutions.

3.3.2 Vacuum Brans–Dicke wormholes [Jordan frame]

For vacuum Brans–Dicke gravity a suitably large class of solutions to the field equations is [28]

$$ds^2 = - \left[\frac{1 - R/r}{1 + R/r} \right]^{2A} dt^2 + [1 + R/r]^4 \left[\frac{1 - R/r}{1 + R/r} \right]^{2+2B} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)] . \quad (34)$$

$$\phi = \phi_0 \left[\frac{1 - B/r}{1 + B/r} \right]^{-(A+B)} . \quad (35)$$

(Note there is a non-propagating typo in equation (8) of [28].) Here we have chosen to work in isotropic coordinates. We have

$$R = \sqrt{\frac{3 + 2\omega}{4 + 2\omega}} \frac{G M}{2} . \quad (36)$$

(This is necessary to get the correct asymptotic behaviour as $r \rightarrow \infty$.) We also have the constraints that

$$\phi_0 = \frac{4 + 2\omega}{3 + 2\omega} . \quad (37)$$

$$A = \sqrt{\frac{4 + 2\omega}{3 + 2\omega}} > 0 . \quad (38)$$

$$B = -\frac{1 + \omega}{2 + \omega} \sqrt{\frac{4 + 2\omega}{3 + 2\omega}} . \quad (39)$$

The metric is real only for $\omega > -3/2$ or $\omega < -2$, the square root always being taken to be positive when it is real, and the metric reduces to the Schwarzschild geometry for $\omega \rightarrow \pm\infty$. The geometry also has a symmetry under $r \rightarrow R^2/r$, but this is potentially misleading. The surface $r = R$ is *not* the throat of a wormhole. To see what is going on, consider the proper circumference of a circle at radius r circumscribing the geometry. we have

$$C(r) = 2\pi r [1 + R/r]^2 \left[\frac{1 - R/r}{1 + R/r} \right]^{1+B} . \quad (40)$$

Assume that R (and hence M) is positive, this can be generalized if desired.

- If $B > -1$ then $C(r) \rightarrow 0$ as $r \rightarrow R$. In this case the surface $r = R$ is a naked curvature singularity as may be verified by direct computation of the curvature invariants. The region $r \in (0, R)$ is a second asymptotically flat region, isomorphic to the first, that connects to the first only at the curvature singularity $r = R$. [$B > -1$ corresponds to $\omega \in (-3/2, +\infty)$.]
- If $B = -1$ then $C(r) \rightarrow 8\pi R$ as $r \rightarrow R$. In this case the surface $r = R$ is at least a surface of finite area. In fact $B = -1$ is achieved only for $\omega = \pm\infty$ in which case the geometry reduces to Schwarzschild. (In this case we also have $A = 1$ and $\phi = \phi_0$.)
- If $B < -1$ then $C(r) \rightarrow \infty$ as $r \rightarrow R$. This corresponds to $\omega \in (-\infty, -2)$. In this case the surface $r = R$ is the second asymptotic spatial infinity associated with a traversable wormhole. The location of the wormhole throat is specified by looking for the minimum value of $C(r)$ which occurs at

$$r_{throat} = R \left[-B + \sqrt{B^2 - 1} \right] > R. \quad (41)$$

The wormhole is in this case *asymmetric* under interchange of the two asymptotic regions ($r = \infty$ and $r = R$). Since $A + B < 0$ in this parameter regime, $\phi \rightarrow 0$ as $r \rightarrow R$, and so the effective Newton constant $G_{eff} = G/\phi$ tends to infinity on the other side of the wormhole. The region near $r = R$ is asymptotically large, but not asymptotically flat, as may be verified by direct computation of the curvature. (It would be interesting to know a little bit more about what this region actually looks like, and to develop a better understanding of the physics on the other side of this class of Brans–Dicke wormholes.) The region $r \in (0, R)$ is now a second completely independent universe, isomorphic to the first, with its own wormhole occurring at

$$r_{throat} = R \left[-B - \sqrt{B^2 - 1} \right] < R. \quad (42)$$

- There is even more parameter space to explore if we look at the extended class of Brans–Dicke solutions discussed in [29], or let the total mass go negative.

In summary, for suitable choices of the parameters, $\omega < -2$, there are wormholes in vacuum Brans–Dicke gravity expressed in the Jordan frame. The null energy condition is still violated, with the Brans–Dicke field providing the exotic matter.

3.3.3 The Einstein Frame

By making a conformal transformation it is possible to express the Brans–Dicke theory in the so-called Einstein frame. The action is now (see [31], modified for Lorentzian signature)

$$S = \int \sqrt{-\tilde{g}} \left\{ \frac{R(\tilde{g})}{2\omega + 3} + \frac{1}{2}(\nabla\sigma)^2 + 16\pi G \exp(2\sigma) \mathcal{L}_{matter} \right\}. \quad (43)$$

Here

$$\tilde{g}_{\mu\nu} = \exp(\sigma) g_{\mu\nu}. \quad (44)$$

$$\phi = \frac{1}{(2\omega + 3)} \exp(\sigma). \quad (45)$$

(And note that the matter Lagrangian, which implicitly depends on the Jordan metric, also has to be carefully rewritten in terms of σ and the Einstein metric.) Now provided ϕ is neither zero nor infinite the conformal transformation from the Jordan to the Einstein frame is globally well-defined. If in addition ϕ goes to a finite non-zero constant at in the asymptotically flat region then it is clear that wormholes, in the sense of topologically nontrivial curves from Scri^- to Scri^+ (past null infinity to future null infinity) can neither be created nor destroyed by a change of frame. (For background see [5] and [3, pages 195–199]). What can, and in general does, change however is the location and even the number of wormhole throats that are encountered in crossing from one asymptotically flat region to another. A wormhole throat as defined by the Einstein frame metric is not necessarily a wormhole throat as defined by the Jordan frame metric.

3.3.4 Vacuum Brans–Dicke wormholes [Einstein frame]

If we look at the vacuum Brans–Dicke field equations in the Einstein frame we get

$$G_{\alpha\beta} = (2\omega + 3) \left\{ \sigma_\alpha \sigma_\beta - \frac{1}{2} g_{\alpha\beta} (\nabla\sigma)^2 \right\}, \quad (46)$$

and

$$\nabla^2 \sigma = 0. \quad (47)$$

In the Einstein frame, the Brans–Dicke field violates the null energy condition only for $\omega < -3/2$, and so vacuum Brans–Dicke wormholes can exist only for $\omega < -3/2$. (This is a necessary but not sufficient condition.) Note that the Jordan and Einstein frames are globally conformally equivalent only if ϕ is never zero. Since the $\omega < -2$ Jordan frame wormholes have $\phi = 0$ at $r = R$ there is a risk of pathological behaviour.

If we take the class of vacuum Brans–Dicke solutions considered previously, and transform them to the Einstein frame we have

$$\begin{aligned}
(2\omega + 4)d\tilde{s}^2 &= - \left[\frac{1 - R/r}{1 + R/r} \right]^{A-B} dt^2 \\
&\quad + [1 + R/r]^4 \left[\frac{1 - R/r}{1 + R/r} \right]^{2+B-A} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)].
\end{aligned} \tag{48}$$

The proper circumference of a circle at radius r circumscribing the geometry is now

$$\tilde{C}(r) = \frac{2\pi r}{\sqrt{|2\omega + 4|}} [1 + R/r]^2 \left[\frac{1 - R/r}{1 + R/r} \right]^{1+(B-A)/2}. \tag{49}$$

Assume again that R (and hence M) is positive. It is now the combination $B - A$ that is of central interest. Indeed

$$A - B = 2\sqrt{\frac{3 + 2\omega}{4 + 2\omega}} > 0. \tag{50}$$

- If $B - A > -2$ then $\tilde{C}(r) \rightarrow 0$ as $r \rightarrow R$. This again corresponds to $\omega \in (-3/2, +\infty)$, and the geometry again contains a naked curvature singularity as may be verified by direct computation of the curvature invariants. The region $r \in (0, B)$ is a second asymptotically flat region, isomorphic to the first, that connects to the first only at the curvature singularity $r = B$.
- If $B - A = 2$ then $\tilde{C}(r) \rightarrow 8\pi R$ as $r \rightarrow R$. In this case the surface $r = R$ is at least a surface of finite area. It is easy to see that $B - A = -2$ implies $\omega = \pm\infty$ and thus $A = 1$, $B = -1$. This again reproduces the Schwarzschild solution.
- If $B - A < -2$ then $\tilde{C}(r) \rightarrow \infty$ as $r \rightarrow R$. The surface $r = R$ is the second asymptotic spatial infinity associated with a traversable wormhole. The location of the wormhole throat is specified by looking for the minimum value of $\tilde{C}(r)$ which occurs at

$$r_{throat} = R \left[\frac{(A - B)}{2} + \sqrt{\frac{(A - B)^2}{4} - 1} \right]. \tag{51}$$

The wormhole is again *asymmetric* under interchange of the two asymptotic regions ($r = \infty$ and $r = R$), and the throat is located at a different place.

- In some sense we have been lucky: The field equations have forced the conformal transform that relates the Jordan and Einstein frames to be sufficiently mild that the two frames agree as to the range of values of the ω parameter that lead to traversable wormhole geometries. There is no a priori necessity for this agreement. (In fact when considering $O(4)$ Euclidean Brans–Dicke wormholes these differences are rather severe [31].) The two frames do differ however in the precise location of the wormhole throat, and in the precise details of the energy condition violations. In the Einstein frame the energy condition violations are obvious from the relative minus sign (for $\omega < -3/2$) in front of the kinetic energy term for the σ field. In the Jordan frame the energy condition violations are still encoded in the Brans–Dicke field (now ϕ) but in a more subtle manner.

In summary, for suitable choices of the parameters ($\omega < -2$), there are wormholes in vacuum Brans–Dicke gravity expressed in the Einstein frame. The null energy condition is still violated, with the Brans–Dicke field providing the exotic matter.

3.3.5 Other exotic wormholes

Similar analyses can be performed for other more or less natural generalizations of Einstein gravity. Dilaton gravity is a particularly important example, inspired by string theory, that is very closely related to Brans–Dicke gravity. Other examples include Einstein–Cartan gravity, Lovelock gravity, Gauss–Bonnet gravity, higher-derivative gravity, etc...

4 Generic static wormholes

We now set aside the special cases we have been discussing, and seek to develop a general analysis of the energy condition violations in static wormholes. (This analysis is based largely on [32]. For an analysis using similar techniques applied to static vacuum and electrovac black holes see Israel [33, 34]. A related decomposition applied to the collapse problem is addressed in [35].) In view of the preceding discussion we want to get away from the notion that topology is the intrinsic defining feature of wormholes and instead focus on the geometry of the wormhole throat. Our strategy is straightforward

- Take any static spacetime, and use the natural time coordinate to slice it into space plus time.
- Use the Gauss–Codazzi and Gauss–Weingarten equations to decompose the $(3 + 1)$ -dimensional spacetime curvature tensor in terms of the 3-dimensional spatial curvature tensor, the extrinsic curvature of the time slices [zero!], and the gravitational potential.

- Take any 3-dimensional spatial slice, and look for a 2-dimensional surface of strictly minimal area. Define such a surface, if it exists, to be the throat of a wormhole. This generalizes the Morris–Thorne flare out condition to arbitrary static wormholes.
- Use the Gauss–Codazzi and Gauss–Weingarten equations again, this time to decompose the 3-dimensional spatial curvature tensor in terms of the 2-dimensional curvature tensor of the throat and the extrinsic curvature of the throat as an embedded hypersurface in the 3-geometry.
- Reassemble the pieces: Write the spacetime curvature in terms of the 2 curvature of the throat, the extrinsic curvature of the throat in 3-space, and the gravitational potential.
- Use the generalized flare-out condition to place constraints on the stress-energy at and near the throat.

4.1 Static spacetimes

In any static spacetime one can decompose the spacetime metric into block diagonal form [36, 37, 38]:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (52)$$

$$= -\exp(2\phi)dt^2 + g_{ij} dx^i dx^j. \quad (53)$$

Notation: Greek indices run from 0–3 and refer to space-time; latin indices from the middle of the alphabet (i, j, k, \dots) run from 1–3 and refer to space; latin indices from the beginning of the alphabet (a, b, c, \dots) will run from 1–2 and will be used to refer to the wormhole throat and directions parallel to the wormhole throat.

Being static tightly constrains the space-time geometry in terms of the three-geometry of space on a constant time slice, and the manner in which this three-geometry is embedded into the spacetime. For example, from [36, page 518] we have

$$^{(3+1)}R_{ijkl} = ^{(3)}R_{ijkl}. \quad (54)$$

$$^{(3+1)}R_{\hat{t}abc} = 0. \quad (55)$$

$$^{(3+1)}R_{\hat{t}i\hat{t}j} = \phi_{|ij} + \phi_{|i} \phi_{|j}. \quad (56)$$

The hat on the t index indicates that we are looking at components in the normalized t direction

$$X_{\hat{t}} = X_t \sqrt{-g^{tt}} = X_t \exp(-\phi). \quad (57)$$

This means we are using an orthonormal basis attached to the fiducial observers (FIDOS). We use $X_{;\alpha}$ to denote a space-time covariant derivative; $X_{|i}$ to denote a three-space covariant derivative, and will shortly use $X_{;a}$ to denote two-space covariant derivatives taken on the wormhole throat itself.

Now taking suitable contractions,

$$^{(3+1)}R_{ij} = ^{(3)}R_{ij} - \phi_{|ij} - \phi_{|i} \phi_{|j}. \quad (58)$$

$$^{(3+1)}R_{\hat{t}\hat{t}} = 0. \quad (59)$$

$$^{(3+1)}R_{\hat{t}\hat{t}} = g^{ij} [\phi_{|ij} + \phi_{|i} \phi_{|j}]. \quad (60)$$

So

$$^{(3+1)}R = ^{(3)}R - 2g^{ij} [\phi_{|ij} + \phi_{|i} \phi_{|j}]. \quad (61)$$

To effect these contractions, we make use of the decomposition of the spacetime metric in terms of the spatial three-metric, the set of vectors e_i^μ tangent to the time-slice, and the vector $V^\mu = \exp[\phi] (\partial/\partial t)^\mu$ normal to the time slice:

$$^{(3+1)}g^{\mu\nu} = e_i^\mu e_j^\nu g^{ij} - V^\mu V^\nu. \quad (62)$$

Finally, for the spacetime Einstein tensor (see [36, page 552])

$$^{(3+1)}G_{ij} = ^{(3)}G_{ij} - \phi_{|ij} - \phi_{|i} \phi_{|j} + g_{ij} g^{kl} [\phi_{|kl} + \phi_{|k} \phi_{|l}]. \quad (63)$$

$$^{(3+1)}G_{\hat{t}\hat{t}} = 0. \quad (64)$$

$$^{(3+1)}G_{\hat{t}\hat{t}} = +\frac{1}{2}^{(3)}R. \quad (65)$$

This decomposition is generic to *any* static spacetime. (You can check this decomposition against various standard textbooks to make sure the coefficients are correct. For instance see Synge [39, page 339], Fock [40], or Adler–Bazin–Schiffer [41])

Observation: Suppose the strong energy condition (SEC) holds then [3]

$$SEC \Rightarrow (\rho + g_{ij}T^{ij}) \geq 0 \quad (66)$$

$$\Rightarrow g^{ij} [\phi_{|ij} + \phi_{|i} \phi_{|j}] \geq 0 \quad (67)$$

$$\Rightarrow \phi \text{ has no isolated maxima.} \quad (68)$$

This is a nice consistency check, and also helpful in understanding the physical import of the strong energy condition.

4.2 Definition of a generic static throat

We define a traversable wormhole throat, Σ , to be a 2-dimensional hypersurface of *minimal* area taken in one of the constant-time spatial slices. Compute the area by taking

$$A(\Sigma) = \int \sqrt{{}^{(2)}g} \, d^2x. \quad (69)$$

Now use Gaussian normal coordinates, $x^i = (x^a; n)$, wherein the hypersurface Σ is taken to lie at $n = 0$, so that

$${}^{(3)}g_{ij} \, dx^i dx^j = {}^{(2)}g_{ab} \, dx^a dx^b + dn^2. \quad (70)$$

The variation in surface area, obtained by pushing the surface $n = 0$ out to $n = \delta n(x)$, is given by the standard computation

$$\delta A(\Sigma) = \int \frac{\partial \sqrt{{}^{(2)}g}}{\partial n} \, \delta n(x) \, d^2x. \quad (71)$$

Which implies

$$\delta A(\Sigma) = \int \sqrt{{}^{(2)}g} \, \frac{1}{2} g^{ab} \frac{\partial g_{ab}}{\partial n} \, \delta n(x) \, d^2x. \quad (72)$$

In Gaussian normal coordinates the extrinsic curvature is simply defined by

$$K_{ab} = -\frac{1}{2} \frac{\partial g_{ab}}{\partial n}. \quad (73)$$

(See [36, page 552]. We use MTW sign conventions. The convention in [3, page 156] is opposite.) Thus

$$\delta A(\Sigma) = - \int \sqrt{{}^{(2)}g} \, \text{tr}(K) \, \delta n(x) \, d^2x. \quad (74)$$

[We use the notation $\text{tr}(X)$ to denote $g^{ab} X_{ab}$.] Since this is to vanish for arbitrary $\delta n(x)$, the condition that the area be *extremal* is simply $\text{tr}(K) = 0$. To force the area to be *minimal* requires (at the very least) the additional constraint $\delta^2 A(\Sigma) \geq 0$. (We shall also consider higher-order constraints below.) But by explicit calculation

$$\delta^2 A(\Sigma) = - \int \sqrt{{}^{(2)}g} \, \left(\frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K)^2 \right) \, \delta n(x) \, \delta n(x) \, d^2x. \quad (75)$$

Extremality [$\text{tr}(K) = 0$] reduces this minimality constraint to

$$\delta^2 A(\Sigma) = - \int \sqrt{{}^{(2)}g} \, \left(\frac{\partial \text{tr}(K)}{\partial n} \right) \, \delta n(x) \, \delta n(x) \, d^2x \geq 0. \quad (76)$$

Since this is to hold for arbitrary $\delta n(x)$ this implies that at the throat we certainly require

$$\frac{\partial \text{tr}(K)}{\partial n} \leq 0. \quad (77)$$

This is the generalization of the Morris–Thorne “flare-out” condition to arbitrary static wormhole throats.

We now invoke some technical fiddles related to the fact that we eventually prefer to have a strong inequality ($<$) at or near the throat, in preference to a weak inequality (\leq) at the throat. Exactly the same type of technical fiddle is required when considering the Morris–Thorne spherically symmetric wormhole. In the following definitions, the two-surface referred to is understood to be embedded in a three-dimensional space, so that the concept of its extrinsic curvature (relative to that embedding space) makes sense.

Definition: Simple flare-out condition.

A two-surface satisfies the “simple flare-out” condition if and only if it is extremal, $\text{tr}(K) = 0$, and also satisfies $\partial \text{tr}(K)/\partial n \leq 0$.

This flare-out condition can be rephrased as follows: We have as an identity that

$$\frac{\partial \text{tr}(K)}{\partial n} = \text{tr} \left(\frac{\partial K}{\partial n} \right) + 2\text{tr}(K^2). \quad (78)$$

So minimality implies

$$\text{tr} \left(\frac{\partial K}{\partial n} \right) + 2\text{tr}(K^2) \leq 0. \quad (79)$$

Generically we would expect the inequality to be strict, in the sense that $\partial \text{tr}(K)/\partial n < 0$, for at least some points on the throat. (See figure 8.) This suggests the modified definition below.

Definition: Strong flare-out condition.

A two surface satisfies the “strong flare-out” condition at the point x if and only if it is extremal, $\text{tr}(K) = 0$, everywhere satisfies $\partial \text{tr}(K)/\partial n \leq 0$, and if at the point x on the surface the inequality is strict:

$$\frac{\partial \text{tr}(K)}{\partial n} < 0. \quad (80)$$

It is sometimes sufficient to demand a weak integrated form of the flare-out condition.

Definition: Weak flare-out condition.

A two surface satisfies the “weak flare-out” condition if and only if it is extremal, $\text{tr}(K) = 0$, and

$$\int \sqrt{{}^{(2)}g} \frac{\partial \text{tr}(K)}{\partial n} d^2x < 0. \quad (81)$$

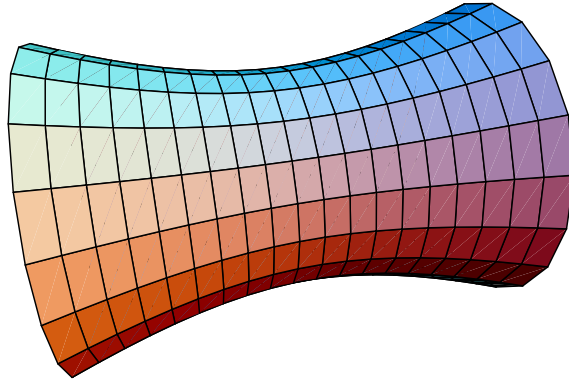


Figure 8: Generically, we define the throat to be located at a true minimum of the area. The geometry should flare-out on either side of the throat, but we make no commitment to the existence of any asymptotically flat region.

Note that the strong flare-out condition implies both the simple flare-out condition and the weak flare-out condition, but that the simple flare-out condition does not necessarily imply the weak flare-out condition. (The integral could be zero.) Whenever we do not specifically specify the type of flare-out condition being used we deem it to be the simple flare-out condition.

The conditions under which the weak definition of flare-out are appropriate arise, for instance, when one takes a Morris–Thorne traversable wormhole (which is symmetric under interchange of the two universes it connects) and distorts the geometry by placing a small bump on the original throat. (See figure 9.)

The presence of the bump causes the old throat to trifurcate into three extremal surfaces: Two minimal surfaces are formed, one on each side of the old throat, (these are minimal in the strong sense previously discussed), while the surface of symmetry between the two universes, though by construction still extremal, is no longer minimal in the strict sense. However, the surface of symmetry is often (but not always) minimal in the weak (integrated) sense indicated above.

A second situation in which the distinction between strong and weak throats is important is in the cut-and-paste construction for traversable wormholes [3, 22, 42]. In this construction one takes two (static) spacetimes $(\mathcal{M}_1, \mathcal{M}_2)$ and excises two geometrically identical regions of the form $\Omega_i \times \mathcal{R}$, Ω_i being compact

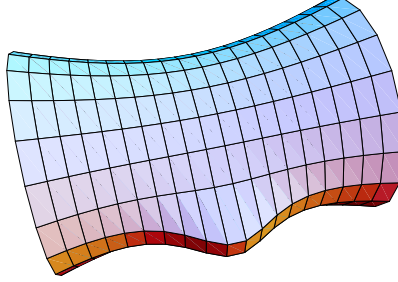


Figure 9: Strong versus weak throats: Placing a small bump on a strong throat typically causes it to tri-furcate into two strong throats plus a weak throat.

spacelike surfaces with boundary and \mathcal{R} indicating the time direction. One then identifies the two boundaries $\partial\Omega_i \times \mathcal{R}$ thereby obtaining a single manifold $(\mathcal{M}_1 \# \mathcal{M}_2)$ that contains a wormhole joining the two regions $\mathcal{M}_i - \Omega_i \times \mathcal{R}$. We would like to interpret the junction $\partial\Omega_{1=2} \times \mathcal{R}$ as the throat of the wormhole.

If the sets Ω_i are convex, then there is absolutely no problem: the junction $\partial\Omega_{1=2} \times \mathcal{R}$ is by construction a wormhole throat in the strong sense enunciated above.

On the other hand, if the Ω_i are concave, then it is straightforward to convince oneself that the junction $\partial\Omega_{1=2} \times \mathcal{R}$ is not a wormhole throat in the strong sense. If one denotes the *convex hull* of Ω_i by $\text{conv}(\Omega_i)$ then the *two* regions $\partial[\text{conv}(\Omega_i)] \times \mathcal{R}$ are wormhole throats in the strong sense. The junction $\partial\Omega_{1=2} \times \mathcal{R}$ is at best a wormhole throat in the weak sense.

For these reasons it is useful to have this notion of a weak throat available as an alternative definition. Whenever we do not qualify the notion of wormhole throat it will refer to a throat in the simple sense. Whenever we refer to a throat in the weak sense or strong senses we will explicitly say so. Finally, it is also useful to define

Definition: Weak f -weighted flare-out condition

A two surface satisfies the “weak f -weighted flare-out” condition if and only if it is extremal, $\text{tr}(K) = 0$, and

$$\int \sqrt{{}^{(2)}g} f(x) \frac{\partial \text{tr}(K)}{\partial n} d^2x < 0. \quad (82)$$

(We will only be interested in this condition for $f(x)$ some positive function defined over the wormhole throat.)

The constraints on the extrinsic curvature embodied in these various definitions lead to constraints on the spacetime geometry, and consequently constraints on the stress-energy.

Technical point I: Degenerate throats

A class of wormholes for which we have to extend these definitions arises when the wormhole throat possesses an accidental degeneracy in the extrinsic curvature at the throat. The previous discussion has tacitly been assuming that near the throat we can write

$$\begin{aligned} {}^{(2)}g_{ab}(x, n) &= {}^{(2)}g_{ab}(x, 0) + n \left. \frac{\partial [{}^{(2)}g_{ab}(x, n)]}{\partial n} \right|_{n=0} \\ &\quad + \frac{n^2}{2} \left. \frac{\partial^2 [{}^{(2)}g_{ab}(x, n)]}{(\partial n)^2} \right|_{n=0} + O[n^3]. \end{aligned} \quad (83)$$

with the linear term having trace zero (to satisfy extremality) and the quadratic term being constrained by the flare-out conditions.

Now if we have an accidental degeneracy with the quadratic (and possibly even higher order terms) vanishing identically, we would have to develop an expansion such as

$$\begin{aligned} {}^{(2)}g_{ab}(x, n) &= {}^{(2)}g_{ab}(x, 0) + n \left. \frac{\partial [{}^{(2)}g_{ab}(x, n)]}{\partial n} \right|_{n=0} \\ &\quad + \frac{n^{2N}}{(2N)!} \left. \frac{\partial^{2N} [{}^{(2)}g_{ab}(x, n)]}{(\partial n)^{2N}} \right|_{n=0} + O[n^{2N+1}]. \end{aligned} \quad (84)$$

Applied to the metric determinant this implies an expansion such as

$$\sqrt{{}^{(2)}g(x, n)} = \sqrt{{}^{(2)}g(x, 0)} \left(1 + \frac{n^{2N}}{(2N)!} k_N(x) + O[n^{2N+1}] \right). \quad (85)$$

Where $k_N(x)$ denotes the first non-zero sub-dominant term in the above expansion, and we know by explicit construction that

$$k_N(x) = +\frac{1}{2} \text{tr} \left(\left. \frac{\partial^{2N} [{}^{(2)}g_{ab}(x, n)]}{(\partial n)^{2N}} \right|_{n=0} \right) \quad (86)$$

$$= -\text{tr} \left(\left. \frac{\partial^{2N-1} K_{ab}(x, n)}{(\partial n)^{2N-1}} \right|_{n=0} \right) \quad (87)$$

$$= - \left(\left. \frac{\partial^{2N-1} K(x, n)}{(\partial n)^{2N-1}} \right|_{n=0} \right), \quad (88)$$

since the trace is taken with $^{(2)}g^{ab}(x, 0)$ and this commutes with the normal derivative. We know that the first non-zero subdominant term in the expansion (85) must be of even order in n (i.e. n^{2N}), and cannot correspond to an odd power of n , since otherwise the throat would be a point of inflection of the area, not a minimum of the area. Furthermore, since $k_N(x)$ is by definition non-zero the flare-out condition must be phrased as the constraint $k_N(x) > 0$, with this now being a strict inequality. More formally, this leads to the definition below.

Definition: N -fold degenerate flare-out condition:

A two surface satisfies the “ N -fold degenerate flare-out” condition at a point x if and only if it is extremal, $\text{tr}(K) = 0$, if in addition the first $2N - 2$ normal derivatives of the trace of the extrinsic curvature vanish at x , and if finally at the point x one has

$$\frac{\partial^{2N-1} \text{tr}(K)}{(\partial n)^{2N-1}} < 0, \quad (89)$$

where the inequality is strict. (In the previous notation this is equivalent to the statement that $k_N(x) > 0$.)

Physically, at an N -fold degenerate point, the wormhole throat is seen to be extremal up to order $2N - 1$ with respect to normal derivatives of the metric, i.e., the flare-out property is delayed spatially with respect to throats in which the flare-out occurs at second order in n . The way we have set things up, the 1-fold degenerate flare-out condition is completely equivalent to the strong flare-out condition.

If we now consider the extrinsic curvature directly we see, by differentiating (85), first that

$$K(x, n) = -\frac{n^{2N-1} k_N(x)}{(2N-1)!} + O[n^{2N}], \quad (90)$$

and secondly that

$$\frac{\partial K(x, n)}{\partial n} = -\frac{n^{2N-2} k_N(x)}{(2N-2)!} + O[n^{2N-1}]. \quad (91)$$

From the dominant $n \rightarrow 0$ behaviour we see that if (at some point x) $2N$ happens to equal 2, then the flare-out condition implies that $\partial K(x, n)/\partial n$ must be negative *at and near the throat*. This can also be deduced directly from the equivalent strong flare-out condition: if $\partial K(x, n)/\partial n$ is negative and non-zero at the throat, then it must remain negative in some region surrounding the throat. On the other hand, if $2N$ is greater than 2 the flare-out condition only tells us that $\partial K(x, n)/\partial n$ must be negative *in some region surrounding the the throat*, and does not necessarily imply that it is negative at the throat itself. (It could merely be zero at the throat.)

Thus for degenerate throats, the flare-out conditions should be rephrased in terms of the first non-zero normal derivative beyond the linear term. Analogous issues arise even for Morris–Thorne wormholes [1, page 405, equation (56)], see also the discussion presented in [3, pages 104–105, 109]. Even if the throat is non-degenerate (1-fold degenerate) there are technical advantages to phrasing the flare-out conditions this way: It allows us to put constraints on the extrinsic curvature near but not on the throat.

Technical point II: Hyperspatial tubes

A second class of wormholes requiring even more technical fiddles arises when there is a central section which is completely uniform and independent of n . [So that $K_{ab} = 0$ over the whole throat for some finite range $n \in (-n_0, +n_0)$.] This central section might be called a “hyperspatial tube”. The flare-out condition should then be rephrased as stating that whenever extrinsic curvature first deviates from zero [at some point $(x, \pm n_0)$] one must formulate constraints such as

$$\left. \frac{\partial \text{tr}(K)}{\partial n} \right|_{\pm n_0^\pm} \leq 0. \quad (92)$$

In this case $\text{tr}(K)$ is by definition not an analytic function of n at n_0 , so the flare-out constraints have to be interpreted in terms of one-sided derivatives in the region outside the hyperspatial tube. [That is, we are concerned with the possibility that $\sqrt{g(x, n)}$ could be constant for $n < n_0$ but behave as $(n - n_0)^{2N}$ for $n > n_0$. In this case derivatives, at $n = n_0$, do not exist beyond order $2N$.]

4.3 Geometry of a generic static throat

Using Gaussian normal coordinates in the region surrounding the throat

$${}^{(3)}R_{abcd} = {}^{(2)}R_{abcd} - (K_{ac}K_{bd} - K_{ad}K_{bc}). \quad (93)$$

See [36, page 514, equation (21.75)]. Because two dimensions is special this reduces to:

$${}^{(3)}R_{abcd} = \frac{{}^{(2)}R}{2} (g_{ac}g_{bd} - g_{ad}g_{bc}) - (K_{ac}K_{bd} - K_{ad}K_{bc}). \quad (94)$$

Of course we still have the standard dimension-independent results that:

$${}^{(3)}R_{nabc} = -(K_{ab;c} - K_{ac;b}). \quad (95)$$

$${}^{(3)}R_{nanb} = \frac{\partial K_{ab}}{\partial n} + (K^2)_{ab}. \quad (96)$$

See [36, page 514, equation (21.76)] and [36, page 516 equation (21.82)]. Here the index n refers to the spatial direction normal to the two-dimensional throat.

Thus far, these results hold both on the throat and in the region surrounding the throat: these results hold as long as the Gaussian normal coordinate system does not break down. (Such breakdown being driven by the fact that the normal geodesics typically intersect after a certain distance.) In the interests of notational tractability we now particularize attention to the throat itself, but shall subsequently indicate that certain of our results can be extended off the throat itself into the entire region over which the Gaussian normal coordinate system holds sway.

Taking suitable contractions, *and using the extremality condition* $\text{tr}(K) = 0$,

$${}^{(3)}R_{ab} = \frac{{}^{(2)}R}{2} g_{ab} + \frac{\partial K_{ab}}{\partial n} + 2(K^2)_{ab}. \quad (97)$$

$${}^{(3)}R_{na} = -K_{ab}{}^{;b}. \quad (98)$$

$${}^{(3)}R_{nn} = \text{tr}\left(\frac{\partial K}{\partial n}\right) + \text{tr}(K^2) \quad (99)$$

$$= \frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2). \quad (100)$$

So that

$${}^{(3)}R = {}^{(2)}R + 2\text{tr}\left(\frac{\partial K}{\partial n}\right) + 3\text{tr}(K^2) \quad (101)$$

$$= {}^{(2)}R + 2\frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2). \quad (102)$$

To effect these contractions, we make use of the decomposition of the three-space metric in terms of the throat two-metric and the set of two vectors e_a^i tangent to the throat and the three-vector n^i normal to the 2-surface

$${}^{(2+1)}g^{ij} = e_a^i e_b^j g^{ab} + n^i n^j. \quad (103)$$

For the three-space Einstein tensor (*cf.* [36, page 552]) we see

$${}^{(3)}G_{ab} = \frac{\partial K_{ab}}{\partial n} + 2(K^2)_{ab} - g_{ab} \frac{\partial \text{tr}(K)}{\partial n} + \frac{1}{2}g_{ab} \text{tr}(K^2). \quad (104)$$

$${}^{(3)}G_{na} = -K_{ab}{}^{;b}. \quad (105)$$

$${}^{(3)}G_{nn} = -\frac{1}{2}{}^{(2)}R - \frac{1}{2}\text{tr}(K^2). \quad (106)$$

Aside: Note in particular that by the flare-out condition ${}^{(3)}R_{nn} \leq 0$. This implies that the three-space Ricci tensor ${}^{(3)}R_{ij}$ has at least one negative semi-definite eigenvalue everywhere on the throat. If we adopt the strong flare-out

condition then the three-space Ricci tensor has at least one negative definite eigenvalue somewhere on the throat. A similar result for Euclidean wormholes is quoted in [43] and the present analysis can of course be carried over to Euclidean signature with appropriate definitional changes.

This decomposition now allows us to write down the various components of the space-time Einstein tensor. For example

$$\begin{aligned}
^{(3+1)}G_{ab} &= -\phi_{|ab} - \phi_{|a} \phi_{|b} + g_{ab} g^{kl} [\phi_{|kl} + \phi_{|k} \phi_{|l}] \\
&\quad + \frac{\partial K_{ab}}{\partial n} + 2(K^2)_{ab} - g_{ab} \frac{\partial \text{tr}(K)}{\partial n} + \frac{1}{2} g_{ab} \text{tr}(K^2) \\
&= 8\pi G T_{ab}.
\end{aligned} \tag{107}$$

But by the definition of the extrinsic curvature, and using the Gauss–Weingarten equations,

$$\phi_{|ab} = \phi_{:ab} + K_{ab} \phi_{|n}. \tag{108}$$

$$\phi_{|na} = K_a{}^b \phi_{:b}. \tag{109}$$

[See, for example, equations (21.57) and (21.63) of [36].] Thus

$$g^{kl} \phi_{|kl} = g^{ab} \phi_{:ab} + (g^{ab} K_{ab}) \phi_{|n} + \phi_{|nn}. \tag{110}$$

But remember that $\text{tr}(K) = 0$ at the throat, so

$$g^{kl} \phi_{|kl} = g^{ab} \phi_{:ab} + \phi_{|nn}. \tag{111}$$

This finally allows us to write

$$\begin{aligned}
^{(3+1)}G_{ab} &= -\phi_{:ab} - \phi_{:a} \phi_{:b} - K_{ab} \phi_{|n} \\
&\quad + g_{ab} [g^{cd} (\phi_{:cd} + \phi_{:c} \phi_{:d}) + \phi_{|nn} + \phi_{|n} \phi_{|n}] \\
&\quad + \frac{\partial K_{ab}}{\partial n} + 2(K^2)_{ab} - g_{ab} \frac{\partial \text{tr}(K)}{\partial n} + \frac{1}{2} g_{ab} \text{tr}(K^2) \\
&= 8\pi G T_{ab}.
\end{aligned} \tag{112}$$

$$\begin{aligned}
^{(3+1)}G_{na} &= -K_a{}^b \phi_{:b} - \phi_{|n} \phi_{:a} - K_{ab}{}^{:b} \\
&= 8\pi G T_{na}.
\end{aligned} \tag{113}$$

$$\begin{aligned}
^{(3+1)}G_{nn} &= g^{cd} [\phi_{:cd} + \phi_{:c} \phi_{:d}] - \frac{1}{2} {}^{(2)}R - \frac{1}{2} \text{tr}(K^2) \\
&= -8\pi G \tau.
\end{aligned} \tag{114}$$

$$^{(3+1)}G_{\hat{t}a} = 0. \tag{115}$$

$$^{(3+1)}G_{\hat{t}n} = 0. \tag{116}$$

$$\begin{aligned}
{}^{(3+1)}G_{\hat{t}\hat{t}} &= \frac{{}^{(2)}R}{2} + \frac{\partial \text{tr}(K)}{\partial n} - \frac{1}{2} \text{tr}(K^2) \\
&= +8\pi G \rho.
\end{aligned} \tag{117}$$

Here τ denotes the *tension* perpendicular to the wormhole throat, it is the natural generalization of the quantity considered by Morris and Thorne, while ρ is simply the energy density at the wormhole throat.

The calculation presented above is in its own way simply a matter of brute force index gymnastics—but we feel that there are times when explicit expressions of this type are useful.

4.4 Constraints on the stress-energy tensor

We can now derive several constraints on the stress-energy.

4.4.1 —First constraint—

$$\tau = \frac{1}{16\pi G} \left[{}^{(2)}R + \text{tr}(K^2) - 2g^{cd}(\phi_{:cd} + \phi_{:c}\phi_{:d}) \right]. \tag{118}$$

(Unfortunately the signs as given are correct. Otherwise we would have a lovely lower bound on τ . We will need to be a little tricky when dealing with the ϕ terms.) The above is the generalization of the Morris–Thorne result that

$$\tau = \frac{1}{8\pi G r_0^2} \tag{119}$$

at the throat of the special class of model wormholes they considered. (With MTW conventions ${}^{(2)}R = 2/r_0^2$ for a two-sphere.) If you now integrate over the surface of the wormhole

$$\int \sqrt{{}^{(2)}g} \tau d^2x = \frac{1}{16\pi G} \left[4\pi\chi + \int \sqrt{{}^{(2)}g} \{ \text{tr}(K^2) - 2g^{cd}\phi_{:c}\phi_{:d} \} d^2x \right]. \tag{120}$$

Here χ is the Euler characteristic of the throat, while the $g^{cd}\phi_{:cd}$ term vanishes by partial integration, since the throat is a manifold without boundary.

4.4.2 —Second constraint—

$$\rho = \frac{1}{16\pi G} \left[{}^{(2)}R + 2\frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2) \right]. \tag{121}$$

The second term is negative semi-definite by the flare-out condition, while the third term is manifestly negative semi-definite. Thus

$$\rho \leq \frac{1}{16\pi G} {}^{(2)}R. \tag{122}$$

This is the generalization of the Morris–Thorne result that

$$\rho = \frac{b'(r_0)}{8\pi G r_0^2} \leq \frac{1}{8\pi G r_0^2} \quad (123)$$

at the throat of the special class of model wormholes they considered. (See [3, page 107].)

Note in particular that if the wormhole throat does not have the topology of a sphere or torus then there *must* be places on the throat such that $^{(2)}R < 0$ and thus such that $\rho < 0$. Thus wormhole throats of high genus will always have regions that violate the weak and dominant energy conditions. (The simple flare-out condition is sufficient for this result. For a general discussion of the energy conditions see [3] or [37].)

If the wormhole throat has the topology of a torus then it will generically violate the weak and dominant energy conditions; only for the very special case $^{(2)}R = 0$, $K_{ab} = 0$, $\partial \text{tr}(K)/\partial n = 0$ will it possibly satisfy (but still be on the verge of violating) the weak and dominant energy conditions. This is a particular example of a degenerate throat in the sense discussed previously.

Wormhole throats with the topology of a sphere will, provided they are convex, at least have positive energy density, but we shall soon see that other energy conditions are typically violated.

If we now integrate over the surface of the wormhole

$$\int \sqrt{{}^{(2)}g} \rho d^2x = \frac{1}{16\pi G} \left[4\pi\chi + \int \sqrt{{}^{(2)}g} \left\{ 2\frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2) \right\} d^2x \right]. \quad (124)$$

So for a throat with the topology of a torus ($\chi = 0$) the simple flare-out condition yields

$$\int \sqrt{{}^{(2)}g} \rho d^2x \leq 0, \quad (125)$$

while the strong or weak flare-out conditions yield

$$\int \sqrt{{}^{(2)}g} \rho d^2x < 0, \quad (126)$$

guaranteeing violation of the weak and dominant energy conditions. For a throat with higher genus topology ($\chi = 2 - 2g$) the simple flare-out condition is sufficient to yield

$$\int \sqrt{{}^{(2)}g} \rho d^2x \leq \frac{\chi}{4G} < 0. \quad (127)$$

4.4.3 —Third constraint—

$$\rho - \tau = \frac{1}{16\pi G} \left[+2 \frac{\partial \text{tr}(K)}{\partial n} - 2\text{tr}(K^2) + 2g^{cd}(\phi_{:cd} + \phi_{:c}\phi_{:d}) \right]. \quad (128)$$

Note that the two-curvature $^{(2)}R$ has conveniently dropped out of this equation. As given, this result is valid only on the throat itself, but we shall soon see that a generalization can be constructed that will also hold in the region surrounding the throat. The first term is negative semi-definite by the simple flare-out condition (at the very worst when integrated over the throat it is negative by the weak flare-out condition). The second term is negative semi-definite by inspection. The third term integrates to zero though it may have either sign locally on the throat. The fourth term is unfortunately positive semi-definite on the throat which prevents us from deriving a truly general energy condition violation theorem without additional information.

Now because the throat is by definition a compact two surface, we know that $\phi(x^a)$ must have a maximum somewhere on the throat. At the global maximum (or even at any local maximum) we have $\phi_{:a} = 0$ and $g^{ab}\phi_{:ab} \leq 0$, so at the maxima of ϕ one has

$$\rho - \tau \leq 0. \quad (129)$$

Generically, the inequality will be strict, and generically there will be points on the throat at which the null energy condition is violated.

Integrating over the throat we have

$$\begin{aligned} & \int \sqrt{{}^{(2)}g} [\rho - \tau] d^2x \\ &= \frac{1}{16\pi G} \int \sqrt{{}^{(2)}g} \left[+2 \frac{\partial \text{tr}(K)}{\partial n} - 2\text{tr}(K^2) + 2g^{cd}(\phi_{:c}\phi_{:d}) \right] d^2x \end{aligned} \quad (130)$$

Because of the last term we must be satisfied with the result

$$\int \sqrt{{}^{(2)}g} [\rho - \tau] d^2x \leq \int \sqrt{{}^{(2)}g} [2g^{cd}(\phi_{:c}\phi_{:d})] d^2x. \quad (131)$$

4.4.4 —Fourth constraint—

We can rewrite the difference $\rho - \tau$ as

$$\rho - \tau = \frac{1}{16\pi G} \left[+2 \frac{\partial \text{tr}(K)}{\partial n} - 2\text{tr}(K^2) + 2\exp(-\phi) {}^{(2)}\Delta \exp(+\phi) \right]. \quad (132)$$

So if we multiply by $\exp(+\phi)$ before integrating, the two-dimensional Laplacian $^{(2)}\Delta$ vanishes by partial integration and we have

$$\begin{aligned}
& \int \sqrt{{}^{(2)}g} \exp(+\phi) [\rho - \tau] d^2x \\
&= \frac{1}{8\pi G} \int \sqrt{{}^{(2)}g} \exp(+\phi) \left[+\frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2) \right] d^2x. \quad (133)
\end{aligned}$$

Thus the strong flare-out condition (or less restrictively, the weak e^ϕ -weighted flare-out condition) implies the violation of this “transverse averaged null energy condition” (TANEC, the NEC averaged over the throat)

$$\int \sqrt{{}^{(2)}g} \exp(+\phi) [\rho - \tau] d^2x < 0. \quad (134)$$

This TANEC, and its off-throat generalization to be developed below, is perhaps the central result of this generic static wormhole analysis.

4.4.5 —Fifth constraint—

We can define an average transverse pressure on the throat by

$$\begin{aligned}
\bar{p} &\equiv \frac{1}{16\pi G} g^{ab} {}^{(3+1)}G_{ab} \quad (135) \\
&= \frac{1}{16\pi G} \left[g^{cd} (\phi_{:cd} + \phi_{:c} \phi_{:d}) + 2\phi_{|nn} + 2\phi_{|n} \phi_{|n} - \frac{\partial \text{tr}(K)}{\partial n} + \text{tr}(K^2) \right]. \quad (136)
\end{aligned}$$

The last term is manifestly positive semi-definite, the penultimate term is positive semi-definite by the flare-out condition. The first and third terms are of indefinite sign while the second and fourth are also positive semi-definite. Integrating over the surface of the throat

$$\int \sqrt{{}^{(2)}g} \bar{p} d^2x \geq \frac{1}{8\pi G} \int \sqrt{{}^{(2)}g} \phi_{|nn} d^2x. \quad (137)$$

A slightly different constraint, also derivable from the above, is

$$\int \sqrt{{}^{(2)}g} e^\phi \bar{p} d^2x \geq \frac{1}{8\pi G} \int \sqrt{{}^{(2)}g} (e^\phi)_{|nn} d^2x. \quad (138)$$

These inequalities relate transverse pressures to normal derivatives of the gravitational potential. In particular, if the throat lies at a minimum of the gravitational red-shift the second normal derivative will be positive, so the transverse pressure (averaged over the wormhole throat) must be positive.

4.4.6 —Sixth constraint—

Now look at the quantities $\rho - \tau + 2\bar{p}$ and $\rho - \tau - 2\bar{p}$. We have

$$\rho - \tau + 2\bar{p} = \frac{1}{4\pi G} \{g^{cd}(\phi_{:cd} + \phi_{:c}\phi_{:d}) + \phi_{|nn} + \phi_{|n}\phi_{|n}\} \quad (139)$$

$$= \frac{1}{4\pi G} \{g^{ij}(\phi_{|ij} + \phi_{|i}\phi_{|j})\}. \quad (140)$$

This serves as a nice consistency check. The combination of stress-energy components appearing above is equal to $\rho + g^{ij}T_{ij}$ and is exactly that relevant to the strong energy condition. See equations (66)–(68). See also equations (63)–(65). Multiplying by e^ϕ and integrating

$$\int \sqrt{{}^{(2)}g} e^\phi [\rho - \tau + 2\bar{p}] d^2x = \frac{1}{4\pi G} \int \sqrt{{}^{(2)}g} (e^\phi)_{|nn} d^2x. \quad (141)$$

This relates this transverse integrated version of the strong energy condition to the normal derivatives of the gravitational potential.

On the other hand

$$\rho - \tau - 2\bar{p} = \frac{1}{4\pi G} \left\{ -\phi_{|nn} - \phi_{|n}\phi_{|n} + \frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2) \right\}. \quad (142)$$

The second and fourth terms are negative semi-definite, while the third term is negative semi-definite by the flare-out condition.

4.4.7 —Summary—

There are a number of powerful constraints that can be placed on the stress-energy tensor at the wormhole throat simply by invoking the minimality properties of the wormhole throat. Depending on the precise form of the assumed flare-out condition, these constraints give the various energy condition violation theorems we are seeking. Even under the weakest assumptions (appropriate to a degenerate throat) they constrain the stress-energy to at best be on the verge of violating the various energy conditions.

4.5 Special case: The isopotential throat

Suppose we take $\phi_{:a} = 0$. This additional constraint corresponds to asserting that the throat is an *isopotential* of the gravitational red-shift. In other words, $\phi(n, x^a)$ is simply a constant on the throat. For instance, all the Morris–Thorne model wormholes [1] possess this symmetry. Under this assumption there are numerous simplifications.

We will not present anew all the results for the Riemann curvature tensor but instead content ourselves with the Einstein tensor

$$\begin{aligned} {}^{(3+1)}G_{ab} &= +g_{ab}(\phi_{|nn} + \phi_{|n}\phi_{|n}) - K_{ab}\phi_{|n} \\ &\quad + \frac{\partial K_{ab}}{\partial n} + 2(K^2)_{ab} - g_{ab}\frac{\partial \text{tr}(K)}{\partial n} + \frac{1}{2}g_{ab}\text{tr}(K^2) \\ &= 8\pi G T_{ab}. \end{aligned} \quad (143)$$

$${}^{(3+1)}G_{na} = -K_{ab}{}^{;b} = 8\pi G T_{na}. \quad (144)$$

$${}^{(3+1)}G_{nn} = -\frac{1}{2}{}^{(2)}R - \frac{1}{2}\text{tr}(K^2) = -8\pi G \tau. \quad (145)$$

$${}^{(3+1)}G_{\hat{t}a} = 0. \quad (146)$$

$${}^{(3+1)}G_{\hat{t}n} = 0. \quad (147)$$

$${}^{(3+1)}G_{\hat{t}\hat{t}} = \frac{{}^{(2)}R}{2} + \frac{\partial \text{tr}(K)}{\partial n} - \frac{1}{2}\text{tr}(K^2) = +8\pi G \rho. \quad (148)$$

Thus for an isopotential throat

$$\tau = \frac{1}{16\pi G} \left[{}^{(2)}R + \text{tr}(K^2) \right] \geq \frac{1}{16\pi G} {}^{(2)}R. \quad (149)$$

$$\rho = \frac{1}{16\pi G} \left[{}^{(2)}R + 2\frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2) \right] \leq \frac{1}{16\pi G} {}^{(2)}R. \quad (150)$$

$$\rho - \tau = \frac{1}{16\pi G} \left[+2\frac{\partial \text{tr}(K)}{\partial n} - 2\text{tr}(K^2) \right] \leq 0. \quad (151)$$

This gives us a very powerful result: using only the simple flare-out condition, the NEC is on the verge of being violated everywhere on an isopotential throat.

By invoking the strong flare-out condition the NEC is definitely violated somewhere on an isopotential throat.

Invoking the weak flare-out condition we can still say that the surface integrated NEC is definitely violated on an isopotential throat.

4.6 Special case: The extrinsically flat throat

Suppose now that we take $K_{ab} = 0$. This is a much stronger constraint than simple minimality of the area of the wormhole throat and corresponds to asserting that the three-geometry of the throat is (at least locally) symmetric under interchange of the two regions it connects. For instance, all the Morris–Thorne model wormholes [1] possess this symmetry and have throats that are extrinsically flat. Under this assumption there are also massive simplifications. (Note that we are not making the isopotential assumption at this stage.)

Again, we will not present all the results but content ourselves with the Einstein tensor

$$\begin{aligned}
{}^{(3+1)}G_{ab} &= -\phi_{:ab} - \phi_{:a} \phi_{:b} + g_{ab} [g^{cd}(\phi_{:cd} + \phi_{:c}\phi_{:d}) + \phi_{|nn} + \phi_{|n}\phi_{|n}] \\
&\quad + \frac{\partial K_{ab}}{\partial n} - g_{ab} \frac{\partial \text{tr}(K)}{\partial n} \\
&= 8\pi G T_{ab}.
\end{aligned} \tag{152}$$

$${}^{(3+1)}G_{na} = -\phi_{|n} \phi_{|a} = 8\pi G T_{na}. \tag{153}$$

$${}^{(3+1)}G_{nn} = g^{cd} [\phi_{:cd} + \phi_{:c}\phi_{:d}] - \frac{1}{2} {}^{(2)}R = -8\pi G \tau. \tag{154}$$

$${}^{(3+1)}G_{\hat{t}a} = 0. \tag{155}$$

$${}^{(3+1)}G_{\hat{t}n} = 0. \tag{156}$$

$${}^{(3+1)}G_{\hat{t}\hat{t}} = \frac{{}^{(2)}R}{2} + \frac{\partial \text{tr}(K)}{\partial n} = +8\pi G \rho. \tag{157}$$

Though the stress-energy tensor is now somewhat simpler than the general case, the presence of the $\phi_{:a}$ terms precludes the derivation of any truly new general theorems.

4.7 Special case: The extrinsically flat isopotential throat

Finally, suppose we take both $K_{ab} = 0$ and $\phi_{:a} = 0$. A wormhole throat that is both extrinsically flat and isopotential is particularly simple to deal with, even though it is still much more general than the Morris–Thorne wormhole. Once again, we will not present all the results but content ourselves with the Einstein tensor

$$\begin{aligned}
{}^{(3+1)}G_{ab} &= +g_{ab} (\phi_{|nn} + \phi_{|n}\phi_{|n}) + \frac{\partial K_{ab}}{\partial n} - g_{ab} \frac{\partial \text{tr}(K)}{\partial n} \\
&= 8\pi G T_{ab}.
\end{aligned} \tag{158}$$

$${}^{(3+1)}G_{na} = 0. \tag{159}$$

$${}^{(3+1)}G_{nn} = -\frac{1}{2} {}^{(2)}R = -8\pi G \tau. \tag{160}$$

$${}^{(3+1)}G_{\hat{t}a} = 0. \tag{161}$$

$${}^{(3+1)}G_{\hat{t}n} = 0. \tag{162}$$

$${}^{(3+1)}G_{\hat{t}\hat{t}} = \frac{{}^{(2)}R}{2} + \frac{\partial \text{tr}(K)}{\partial n} = +8\pi G \rho. \tag{163}$$

In this case $\rho - \tau$ is particularly simple:

$$\rho - \tau = \frac{1}{8\pi G} \frac{\partial \text{tr}(K)}{\partial n}. \tag{164}$$

This quantity is manifestly negative semi-definite by the simple flare-out condition.

- For the strong flare-out condition we deduce that the NEC must be violated somewhere on the wormhole throat.
- Even for the weak flare-out condition we have

$$\int \sqrt{{}^{(2)}g} [\rho - \tau] d^2x < 0. \quad (165)$$

- We again see that generic violations of the null energy condition are the rule.

4.8 The region surrounding the throat

Because the spacetime is static, one can unambiguously define the energy density everywhere in the spacetime by setting

$$\rho = \frac{{}^{(3+1)}G_{\hat{t}\hat{t}}}{8\pi G}. \quad (166)$$

The normal tension, which we have so far defined only on the wormhole throat itself, can meaningfully be extended to the entire region where the Gaussian normal coordinate system is well defined by setting

$$\tau = -\frac{{}^{(3+1)}G_{nn}}{8\pi G}. \quad (167)$$

Thus in particular

$$\rho - \tau = \frac{{}^{(3+1)}G_{\hat{t}\hat{t}} + {}^{(3+1)}G_{nn}}{8\pi G} = \frac{{}^{(3+1)}R_{\hat{t}\hat{t}} + {}^{(3+1)}R_{nn}}{8\pi G}, \quad (168)$$

with this quantity being well defined throughout the Gaussian normal coordinate patch. (The last equality uses the fact that $g_{\hat{t}\hat{t}} = -1$ while $g_{nn} = +1$.) But we have already seen how to evaluate these components of the Ricci tensor. Indeed

$${}^{(3+1)}R_{\hat{t}\hat{t}} = g^{ij} [\phi_{|ij} + \phi_{|i}\phi_{|j}]. \quad (169)$$

$${}^{(3+1)}R_{nn} = {}^{(3)}R_{nn} - [\phi_{|nn} + \phi_{|n}\phi_{|n}] \quad (170)$$

$$= \frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2) - [\phi_{|nn} + \phi_{|n}\phi_{|n}], \quad (171)$$

where we have been careful to *not* use the extremality condition $\text{tr}(K) = 0$. Therefore

$$\begin{aligned}
\rho - \tau &= \frac{1}{8\pi G} \left[\frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2) + g^{ab} (\phi_{|ab} + \phi_{|a} \phi_{|b}) \right] \\
&= \frac{1}{8\pi G} \left[\frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2) + \text{tr}(K) \phi_{|n} + g^{ab} (\phi_{:ab} + \phi_{:a} \phi_{:b}) \right],
\end{aligned} \tag{172}$$

(173)

where in the last line we have used the Gauss–Weingarten equations.

- If the throat is *isopotential*, where isopotential now means that near the throat the surfaces of constant gravitational potential coincide with the surfaces of fixed n , this simplifies to:

$$\rho - \tau = \frac{1}{8\pi G} \left[\frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2) + \text{tr}(K) \phi_{|n} \right]. \tag{174}$$

- If the throat is non-degenerate and satisfies the simple flare-out condition, then at the throat the first and second terms are negative semi-definite, and the third is zero. Then the null energy condition is either violated or on the verge of being violated at the throat.
- If the throat is non-degenerate and satisfies the strong flare-out condition at the point x , then the first term is negative definite, the second is negative semi-definite, and the third is zero. Then the null energy condition is violated at the point x on the throat.
- If the throat satisfies the N -fold degenerate flare-out condition at the point x , then by the generalization of the flare-out conditions applied to degenerate throats the first term will be $O[n^{2N-2}]$ and negative definite in some region surrounding the throat. The second term is again negative semi-definite. The third term can have either sign but will be $O[n^{2N-1}]$. Thus there will be some region $n \in (0, n_*)$ in which the first term dominates. Therefore the null energy condition is violated along the line $\{x\} \times (0, n_*)$. If at every point x on the throat the N -fold degenerate flare-out condition is satisfied for some *finite* N , then there will be an open region surrounding the throat on which the null energy condition is everywhere violated.
- This is the closest one can get in generalizing to arbitrary worm-hole shapes the discussion on page 405 [equation (56)] of Morris–Thorne [1]. Note carefully their use of the phrase “at or near the throat”. In our parlance, they are considering a spherically symmetric extrinsically flat isopotential throat that satisfies the N -fold degenerate flare-out condition for some finite but unspecified N . See also page 104, equation (11.12) and page 109, equation (11.54) of [3], and contrast this with equation (11.56).

- If the throat is not isopotential we multiply by $\exp(\phi)$ and integrate over surfaces of constant n . Then

$$\begin{aligned} \int \sqrt{{}^{(2)}g} \exp(\phi) [\rho - \tau] d^2x = \\ \frac{1}{8\pi G} \int \sqrt{{}^{(2)}g} \exp(\phi) \left[\frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2) + \text{tr}(K)\phi_{|n} \right] d^2x. \end{aligned} \quad (175)$$

This generalizes the previous version (133) of the transverse averaged null energy condition to constant n hypersurfaces near the throat. For each point x on the throat, assuming the N -fold degenerate flare-out condition, we can by the previous argument find a range of values $[n \in (0, n_*(x))]$ that will make the integrand negative. Thus there will be a set of values of n for which the integral is negative. Again we deduce violations of the null energy condition.

5 Discussion

In this survey we have sought to give an overview of the energy condition violations that occur in traversable wormholes. We point out that in static spherically symmetric geometries these violations of the energy conditions follow unavoidably from the definition of a wormhole and the definition of the total stress-energy via the Einstein equations. In spherically symmetric time dependent situations limited temporary suspensions of the energy condition violations are possible. In non-Einstein theories of gravity it is often possible to push the energy condition violations into the nonstandard parts of the stress-energy tensor and let the ordinary part of the stress-energy satisfy the energy conditions. The total stress-energy tensor, however, must still violate the energy conditions.

To show the generality of the energy condition violations, we have developed an analysis that is capable of dealing with static traversable wormholes of arbitrary shape. We have presented a definition of a wormhole throat that is much more general than that of the Morris–Thorne wormhole [1]. The present definition works well in any static spacetime and nicely captures the essence of the idea of what we would want to call a wormhole throat.

We do not need to make any assumptions about the existence of any asymptotically flat region, nor do we need to assume that the manifold is topologically non-trivial. It is important to realise that the essence of the definition lies in the geometrical structure of the wormhole throat.

Starting from our definition we have used the theory of embedded hypersurfaces to place restrictions on the Riemann tensor and stress-energy tensor at the throat of the wormhole. We find, as expected, that the wormhole throat generically violates the null energy condition and we have provided several theorems

regarding this matter. These theorems generalize the Morris–Thorne results on exotic matter [1], and are complementary to the topological censorship theorem [5].

Generalization to the time dependent situation is clearly of interest. Unfortunately we have encountered many subtleties of definition, notation, and formalism in this endeavor. (A formulation in terms of *anti-trapped surfaces* appears promising [6].) We defer the issue of time dependent wormhole throats to a future publication.

Acknowledgements

M.V. wishes to gratefully acknowledge the hospitality shown during his visits to the Laboratory for Space Astrophysics and Fundamental Physics (LAEFF, Madrid). This work was supported in part by the US Department of Energy (M.V.) and by the Spanish Ministry of Science and Education (D.H.).

References

- [1] M.S. Morris and K.S. Thorne, Am. J. Phys. **56**, 395 (1988).
- [2] M.S. Morris, K.S. Thorne and U. Yurtsever, Phys. Rev. Lett, **61**, 1446 (1988).
- [3] M. Visser, *Lorentzian Wormholes: From Einstein to Hawking* (American Institute of Physics, Woodbury, N.Y., 1995).
- [4] D. Hochberg, A. Popov and S.V. Sushkov, Phys. Rev. Lett. **78**, 2050 (1997).
- [5] J.L. Friedmann, K. Schleich and D.M. Witt, Phys. Rev. Lett. **71**, 1486 (1993).
- [6] D. Page, cited as a note added in proof in [1].
- [7] V. Frolov and I. D. Novikov Phys. Rev. D48 (1993) 1607-1615.
- [8] J. A. Wheeler, Phys. Rev. 97 (1955) 511-536.
- [9] J. A. Wheeler, Ann. Phys. (NY) 2 (1957) 604–616.
- [10] S. W. Hawking, Phys. Rev. D46 (1992) 603–611.
- [11] M. Visser, *The reliability horizon for semiclassical quantum gravity: Metric fluctuations are often more important than back reaction*. gr-qc/9702041; Physics Letters B, in press.
- [12] M. Visser, gr-qc/9409043, Phys. Lett. B349 (1995) 443–447.

- [13] M. Visser, gr-qc/9604007, Phys. Rev. D54 (1996) 5103–5115.
- [14] M. Visser gr-qc/9604008, Phys. Rev. D54 (1996) 5116–5122.
- [15] M. Visser, gr-qc/9604009, Phys. Rev. D54 (1996) 5123–5128.
- [16] M. Visser, gr-qc/9703001, Phys. Rev. D56 (1997) 936–952.
- [17] M. Visser, Phys. Rev. D43 (1991) 402–409.
- [18] M. Visser, Phys. Lett. B242 (1990) 24–28.
- [19] M. Visser, *Quantum wormholes in Lorentzian signature*, in B. Bonnor and H. Miettinen, editors, *Proceedings of the Rice meeting: 1990 meeting of the Division of Particles and Fields of the American Physical Society*, volume 2, pages 858–860. (World Scientific, Singapore, 1990).
- [20] F. Schein and P.C. Aichelburg, gr-qc/9606069, Phys. Rev. Lett. 77 (1996) 4130–4133.
- [21] F. Schein, P.C. Aichelburg, and W. Israel, gr-qc/9602053, Phys. Rev. D54 (1996) 3800–3805.
- [22] M. Visser, Phys. Rev. D39, 3182 (1989).
- [23] T. A. Roman, Phys. Rev. D47 (1993) 1370–1379.
- [24] D. Hochberg and T. W. Kephart, Phys. Rev. Lett 70 (1993) 2665–2668.
- [25] S. Kar, Phys. Rev. D49 (1994) 862–865.
- [26] S. Kar and D. Sahdev, Phys. Rev. D53 (1996) 722–730.
- [27] S. W. Kim, Phys. Rev. D53 (1996) 6889–6892.
- [28] A. G. Agnese and M. La Camera, Phys. Rev. D51 (1995) 2011–2013.
- [29] K. K. Nandi, A. Islam, and J. Evans, Phys. Rev. D55 (1997) 2497–2450.
- [30] L. A. Anchordoqui, S. Perez Bergliaffa, and D. F. Torres, Phys. Rev. D55 (1997) 5226–5229.
- [31] H. H. Yang and Y. Z. Zhang, Phys. Lett. A 212 (1996) 39–42.
- [32] D. Hochberg and M. Visser, *Geometric structure of the generic static traversable wormhole throat*, gr-qc/9704082; Phys. Rev. D (in press).
- [33] W. Israel, Phys. Rev. 164 (1967) 1776–1779.
- [34] W. Israel, Commun. Math. Phys. 8 (1968) 254–260.

- [35] W. Israel, *Can. J. Phys.* **64** (1986) 120–127.
- [36] C.W. Misner, K.S. Thorne and J.A. Wheeler *Gravitation* (W.H. Freeman, San Francisco, 1973).
- [37] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).
- [38] R.M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
- [39] J.L. Synge, *Relativity: the General Theory* (North-Holland, Amsterdam, 1964).
- [40] V. Fock, *The Theory of Space, Time, and Gravitation* (Pergamon, New York, 1964).
- [41] R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw–Hill, New York, 1965).
- [42] M. Visser, *Nucl. Phys.* **B328**, 203 (1989).
- [43] S. Giddings and A. Strominger, *Nucl. Phys.* **B306** (1988) 890–907, see esp. page 894.